INVARIANT METRICS ON NEGATIVELY PINCHED COMPLETE KÄHLER MANIFOLDS

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Abstract. We prove that a complete Kähler manifold with holomorphic curvature bounded between two negative constants admits a unique complete Kähler-Einstein metric. We also show this metric and the Kobayashi-Royden metric are both uniformly equivalent to the background Kähler metric. Furthermore, all three metrics are shown to be uniformly equivalent to the Bergman metric, if the complete Kähler manifold is simply-connected, with the sectional curvature bounded between two negative constants. In particular, we confirm two conjectures of R. E. Greene and H. Wu posted in 1979.

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1. Introduction

The classical Liouville's theorem tells us that the complex plane $\mathbb{C}$ has no bounded nonconstant holomorphic functions, while, by contrast, the unit disk $\mathbb{D}$ has plenty of bounded nonconstant holomorphic functions. From a geometric viewpoint, the complex plane does not admit any metric of negative bounded-away-from-zero curvature, while the unit disk admits a metric, the Poincaré metric, of constant negative curvature.

The first author was partially supported by the NSF grant DMS-1611745. The second author was partially supported by the NSF grant DMS-1308244 and DMS-1607871. The first author would like to thank Professor H. Wu for bringing his attention to the Bergman metric in 2006.

In a higher dimensional analogue, the unit disk is replaced by the simply-connected complete Kähler manifold. It is believed that a simply-connected complete Kähler manifold M with sectional curvature bounded above by a negative constant has many nonconstant bounded holomorphic functions (cf. [Yau82, p. 678, Problem 38]). In fact, it is conjectured that such a manifold is biholomorphic to a bounded domain in $\mathbb{C}^n$ (cf. [SY77, p. 225], [Wu83, p. 98]; see also [Wu67, p. 195, (1)] for a related problem with regard to the holomorphic sectional curvature).

The negatively curved complex manifolds are naturally associated with the invariant metrics. An *invariant metric* is a metric L_M defined on a complex manifold M such that every biholomorphism F from M to itself gives an isometry $F^*L_M = L_M$. Thus, the invariant metric depends only on the underlying complex structure of M.

There are four classical invariant metrics, the Bergman metric, the Carathéodory-Reiffen metric, the Kobayashi-Royden metric, and the complete Kähler-Einstein metric of negative scalar curvature. It is known that on a bounded, smooth, strictly pseudoconvex domain in $\mathbb{C}^n$, all four classical invariant metrics are uniformly equivalent to each other (see, for example, [Die70, Gra75, CY80, Lem81, BFG83, Wu93] and references therein). The equivalences do not extended to weakly pseudoconvex domains (see, for example, [DFH84] and references therein for the inequivalence of the Bergman metric and the Kobayashi-Royden metric).

On Kähler manifolds, R. E. Greene and H. Wu have posted two remarkable conjectures concerning the uniform equivalences of the Kobayashi-Royden metric and the Bergman metric. Their first conjecture states as below.

Conjecture 1 ([GW79, p. 112, Remark (2)])**.** *Let (M, ω) be a simply-connected complete Kähler manifold satisfying $-B \leq$ sectional curvature $\leq -A$ for two positive constants A and B. Then, the Kobayashi-Royden metric $\mathfrak{K}$ satisfies*

$$C^{-1}|\xi|_\omega \leq \mathfrak{K}(x,\xi) \leq C|\xi|_\omega, \quad \text{for all } x \in M \text{ and } \xi \in T'_xM.$$

Here $C > 0$ is a constant depending only on A and B.

As pointed out in [GW79, p. 112], it is well-known that the left inequality in the conjecture follows from the Schwarz lemma and the hypothesis of sectional curvature bounded above by a negative constant (see also Lemma 19).

Our first result confirms this conjecture. In fact, we prove a stronger result, as we relax the sectional curvature to the holomorphic sectional curvature, and remove the assumption of simply-connectedness.

Theorem 2. *Let (M, ω) be a complete Kähler manifold whose holomorphic sectional curvature $H(\omega)$ satisfies $-B \leq H(\omega) \leq -A$ for some positive constants A and B. Then, the Kobayashi-Royden metric $\mathfrak{K}$ satisfies*

$$C^{-1}|\xi|_\omega \leq \mathfrak{K}(x,\xi) \leq C|\xi|_\omega, \quad \text{for all } x \in M \text{ and } \xi \in T'_xM.$$

Here $C > 0$ is a constant depending only on A, B and $\dim M$.

Under the same condition as Theorem 2, we construct a unique complete Kähler-Einstein metrics of negative Ricci curvature, and show that it is uniformly equivalent to the background Kähler metric.

Theorem 3. *Let (M,ω) be a complete Kähler manifold whose holomorphic sectional curvature $H(\omega)$ satisfies $-\kappa_2 \leq H(\omega) \leq -\kappa_1$ for constants $\kappa_1, \kappa_2 > 0$. Then M admits a unique complete Kähler-Einstein metric ω_{KE} with Ricci curvature equal to -1, satisfying*

$$C^{-1}\omega \leq \omega_{\mathrm{KE}} \leq C\omega \quad \text{on } M$$

for some constant $C > 0$ depending only on $\dim M$, κ_1 and κ_2. Furthermore, the curvature tensor $R_{\mathrm{m,KE}}$ of ω_{KE} and all its covariant derivatives are bounded; that is, for each $l \in \mathbb{N}$,

$$\sup_{x\in M} \left|\nabla^l R_{\mathrm{m,KE}}(x)\right|_{\omega_{\mathrm{KE}}} \leq C_l$$

where $C_l > 0$ depends only on l, $\dim M$, κ_1, and κ_2.

Theorem 3 differs from the previous work on complete noncompact Kähler-Einstein metrics such as [CY80, CY86, TY87, Wu08] in that we put no assumption on the sign of Ricci curvature $\mathrm{Ric}(\omega)$ of metric ω, nor on $\mathrm{Ric}(\omega) - \omega$. The proof makes use of a new complex Monge-Ampère type equation, which involves the Kähler class of $t\omega - \mathrm{Ric}(\omega)$ rather than that of ω. This equation is inspired by our recent work [WY16a]. Theorem 3 can be viewed as a complete noncompact generalization of [WY16a, Theorem 2] (for its generalizations on compact manifolds, see for example [TY17, DT19, WY16b, YZ19].)

We now discuss the second conjecture of Greene-Wu concerning the Bergman metric. Greene-Wu has obtained the following result, motivated by the work of the second author and Y. T. Siu [SY77].

Theorem 4 ([GW79, p. 144, Theorem H (3)])**.** *Let (M,ω) be a simply-connected complete Kähler manifold such that $-B \leq$ sectional curvature $\leq -A$ for some positive constants A and B. Then, M possesses a complete Bergman metric $\omega_{\mathfrak{B}}$ satisfying*

$$\omega_{\mathfrak{B}} \geq C\omega \quad \text{on } M,$$

for some constant $C > 0$ depending only on $\dim M$, A, and B. Moreover, the Bergman kernel form $\mathfrak{B}$ on M satisfies

$$A_1\omega^n \leq \mathfrak{B} \leq A_2\omega^n \quad \text{on } M, \tag{1.1}$$

for some positive constants A_1, A_2 depending only on $\dim M$, A, and B.

It is shown in [GW79, p. 144, Theorem H (2)] that, if a simply-connected complete Kähler manifold M satisfies $-B/r^2 \leq$ sectional curvature $\leq -A/r^2$ outside a compact subset of M, then M possesses a complete Bergman metric, where r is the distance from a fixed point. Greene-Wu proposed two conjectures concerning their Theorem H. The first conjecture is that the lower bound $-B/r^2$ in the hypothesis of Theorem H (2) can be removed. This has been settled by B. Y. Chen and J. H. Zhang [CZ02]. The second conjecture is as below.

Conjecture 5 ([GW79, p. 145, Remark (3)]). *The Bergman metric $\omega_{\mathfrak{B}}$ obtained in Theorem 4 satisfies*

$$\omega_{\mathfrak{B}} \le C_1 \omega \quad \text{on } M$$

for some constant $C_1 > 0$. As a consequence, the Bergman metric $\omega_{\mathfrak{B}}$ is uniformly equivalent to the background Kähler metric ω.

Conjecture 5 now follows from the following result.

Theorem 6. *Let (M, ω) be a complete, simply-connected, Kähler manifold such that $-B \le \text{sectional curvature} \le -A < 0$ for some positive constants A and B. Then, Bergman metric $\omega_{\mathfrak{B}}$ has bounded geometry, and satisfies*

$$\omega_{\mathfrak{B}} \le C_1 \omega \quad \text{on } M,$$

where the constant $C_1 > 0$ depending only on A, B, and $\dim M$. As a consequence, the Bergman metric $\omega_{\mathfrak{B}}$ is uniformly equivalent to the Kähler metric ω.

The simple connectedness assumption is necessary for the equivalence of $\omega_{\mathfrak{B}}$ and ω. For example, let $M = \mathbb{P}^1 \setminus \{0, 1, \infty\}$. Then, M has a complete Kähler-Einstein metric with curvature equal to -1; however, M admits no Bergman metric. Another example is the punctured disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ together with the complete Poincaré metric $\omega_{\mathcal{P}} = \sqrt{-1} dz \wedge d\bar{z}/(|z| \log |z|^2)^2$. Note that the Bergman metric on $\mathbb{D}^*$ is $\omega_{\mathfrak{B}} = \sqrt{-1} dz \wedge d\bar{z}/(1 - |z|^2)^2$, which cannot dominate $\omega_{\mathcal{P}}$ at the origin.

The equivalence of $\omega_{\mathfrak{B}}$ and ω in Theorem 6 has been known in several cases: For instance, when M is a bounded strictly pseudoconvex domain with smooth boundary, this can be shown by using the asymptotic expansion of Monge-Ampère equation (see [BFG83] for example). The second author with K. Liu and X. Sun [LSY04] has proved the result for M being the Teichmüller space and the moduli space of Riemann surfaces, on which they in fact show that several classical and new metrics are all uniformly equivalent (see also [Yeu05]); compare Corollary 7 below.

As a consequence of the above theorems, we obtain the following result on a complete, simply-connected, Kähler manifold with negatively pinched sectional curvature.

Corollary 7. *Let (M, ω) be a complete, simply-connected, Kähler manifold satisfying $-B \le \text{sectional curvature} \le -A$ for two positive constants A and B. Then, the Kähler-Einstein metric ω_{KE}, the Bergman metric $\omega_{\mathfrak{B}}$, and the Kobayashi-Royden metric $\mathfrak{K}$ all exist, and are all uniformly equivalent to ω on M, where the equivalence constants depend only on A, B, and $\dim M$.*

Corollary 7 in particular implies that the smoothly bounded weakly pseudoconvex domain Ω constructed in [DFH84] and [JP13, p. 491], given by

$$\Omega = \{(z_1, z_2, z_3) \in \mathbb{C}^3; \mathrm{Re} z_1 + |z_1|^2 + |z_2|^{12} + |z_3|^{12} + |z_2|^4 |z_3|^2 + |z_2|^2 |z_3|^6 < 0\},$$

cannot admit a complete Kähler metric with negative pinched sectional curvature.

In this paper we provide a unifying treatment for the invariant metrics, through developing the techniques of effective quasi-bounded geometry. The quasi-bounded geometry was originally introduced to solve the Monge-Ampère equation on the complete noncompact manifold with injectivity radius zero. By contrast to solving equations, the holomorphicity of quasi-coordinate map is essential for our applications to invariant metrics. It is crucial to show the radius of quasi-bounded geometry depends only on the curvature bounds. Then, a key ingredient is the *pointwise interior estimate*. Several arguments, such as Lemma 12, Lemma 15, Lemma 20, and Corollary 24, may have interests of their own.

Notation and Convention. We interchangeably denote a hermitian metric by tensor $g_\omega = \sum_{i,j} g_{i\bar{j}} dz^i \otimes d\bar{z}^j$ and its Kähler form $\omega = (\sqrt{-1}/2)\sum_{i,j} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$. The curvature tensor $R_{\mathrm{m}} = \{R_{i\bar{j}k\bar{l}}\}$ of ω is given by

$$R_{i\bar{j}k\bar{l}} = R\Big(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{z}^j}, \frac{\partial}{\partial z^k}, \frac{\partial}{\partial \bar{z}^l}\Big) = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^l} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{i\bar{q}}}{\partial z^k} \frac{\partial g_{p\bar{j}}}{\partial \bar{z}^l}.$$

Let x be a point in M and $\eta \in T'_x M$ be a unit holomorphic tangent vector at x. Then, the holomorphic (sectional) curvature of ω at x in the direction η is

$$H(\omega, x, \eta) = H(x, \eta) = R(\eta, \overline{\eta}, \eta, \overline{\eta}) = \sum_{i,j,k,l} R_{i\bar{j}k\bar{l}} \eta^i \bar{\eta}^j \eta^k \bar{\eta}^l.$$

We abbreviate $H(\omega) \leq \kappa$ (resp. $H(\omega) \geq \kappa$) for some constant κ, if $H(\omega, x, \eta) \leq \kappa$ (resp. $H(\omega, x, \eta) \geq \kappa$) at every point x of M and for each $\eta \in T'_x M$. We denote

$$dd^c \log \omega^n = dd^c \log \det(g_{i\bar{j}}) = \frac{\sqrt{-1}}{2} \partial\bar{\partial} \log \det(g_{i\bar{j}}) = -\mathrm{Ric}(\omega)$$

where $d^c = \sqrt{-1}(\bar{\partial} - \partial)/4$.

We say that two pseudometrics L_1 and L_2 are *uniformly equivalent* or *quasi-isometric* on a complex manifold M, if there exists a constant $C > 0$ such that

$$C^{-1} L_1(x, \xi) \leq L_2(x, \xi) \leq C L_1(x, \xi) \quad \text{for all } x \in M,\ \xi \in T'_x M,$$

which is often abbreviated as $C^{-1} L_1 \leq L_2 \leq C L_1$ on M.

In many estimates, we give quite explicit constants mainly to indicate their dependence on the parameters such as $\dim M$ and the curvature bounds.

2. Effective quasi-bounded geometry

The notions of *bounded geometry* and *quasi-bounded geometry* are introduced by the second author and S. Y. Cheng, originally to adapt the Schauder type estimates to solve the Monge-Ampère type equation on complete noncompact manifolds (see, for example, [Yau78b], [CY80, CY86], [TY87, TY90, TY91] and [WL97, Appendix]).

We use the following the formulation (compare [TY90, p. 580] for example). Let (M, ω) be an n-dimensional complete Kähler manifold. For a point $P \in M$, let $B_\omega(P; \rho)$ be the open geodesic ball centered at P in M of radius ρ; sometimes we omit the subscript ω when there is no confusion. Denote by $B_{\mathbb{C}^n}(0; r)$ the open ball centered at the origin in $\mathbb{C}^n$ of radius r with respect to the standard metric $\omega_{\mathbb{C}^n}$.

Definition 8. An n-dimensional Kähler manifold (M, ω) is said to have *quasi-bounded geometry*, if there exist two constants $r_2 > r_1 > 0$, such that for each point P of M, there is a domain U in $\mathbb{C}^n$ and a nonsingular holomorphic map $\psi : U \to M$ satisfying the following properties

(i) $B_{\mathbb{C}^n}(0; r_1) \subset U \subset B_{\mathbb{C}^n}(0; r_2)$ and $\psi(0) = P$;

(ii) there exists a constant $C > 0$ depending only on r_1, r_2, n such that

$$C^{-1}\omega_{\mathbb{C}^n} \le \psi^*\omega \le C\omega_{\mathbb{C}^n} \quad \text{on } U; \tag{2.1}$$

(iii) for each integer $l \ge 0$, there exists a constant A_l depending only on l, n, r_1, r_2 such that

$$\sup_{x \in U} \left| \frac{\partial^{|\nu|+|\mu|} g_{i\bar{j}}}{\partial v^\mu \partial \bar{v}^\nu}(x) \right| \le A_l, \quad \text{for all } |\mu| + |\nu| \le l \tag{2.2}$$

where $g_{i\bar{j}}$ is the component of $\psi^*\omega$ on U in terms of the natural coordinates $(v^1, \dots, v^n)$, and μ, ν are the multiple indices with $|\mu| = \mu_1 + \cdots + \mu_n$.

The map ψ is called a *quasi-coordinate map* and the pair (U, ψ) is called a *quasi-coordinate chart* of M. We call the positive number r_1 a *radius of quasi-bounded geometry*. The Kähler manifold (M, ω) is of *bounded geometry* if in addition each $\psi : U \to M$ is biholomorphic onto its image. In this case, the number r_1 is called *radius of bounded geometry*.

The following theorem is fundamental on constructing the quasi-coordinate charts.

Theorem 9. *Let (M, ω) be a complete Kähler manifold.*

(1) The manifold (M, ω) has quasi-bounded geometry if and only if for each integer $q \ge 0$, there exists a constant $C_q > 0$ such that

$$\sup_{P \in M} |\nabla^q R_{\mathrm{m}}| \le C_q, \tag{2.3}$$

where $R_{\mathrm{m}} = \{R_{i\bar{j}k\bar{l}}\}$ denotes the curvature tensor of ω. In this case, the radius of quasi-bounded geometry depends only on C_0 and $\dim M$.

(2) If (M, ω) has positive injectivity radius and the curvature tensor R_{m} of ω satisfies (2.3), then (M, ω) has bounded geometry. The radius of bounded geometry depends only on C_0, $\dim M$, and also the injectivity radius r_ω of ω unless r_ω is infinity.

Theorem 9 (1) is especially useful for a complete Kähler manifold with injectivity radius zero. Compare, for example, [TY87, pp. 602–605], [Wu08], [GW16] for the explicit construction of quasi-bounded geometry on the quasi-projective manifolds; compare also [Shi97, p. 212, Lemma 9.2] for the construction using the Ricci flow, under the additional assumption of positive bisectional curvature.

The Riemannian version of quasi-bounded geometry is constructed by the second author in 1980 (cf. [WL97, Appendix]), by taking the exponential map $\exp_P$ as the smooth quasi-coordinate map, with its domain U being the ball $B(0; R)$ in the tangent space at P. The radius R can be chosen to depend only on the upper bound of the curvature. The real quasi-bounded geometry is sufficient to adapt the Schauder estimates to solve equations on a Riemann manifold, regardless of its injectivity radius.

One can solve the holomorphic functions $\{v^j\}$ out of the normal coordinates on $B(0;R)$, by using the L^2-estimate of $\bar\partial$-operator. By Siu-Yau's inequality, one applies the singular L^2-weight to ensure that $\{dv^j\}$ are independent at P. Then, the holomorphic functions $\{v^j\}$ form a coordinate system in a smaller ball $B(0;r)$, by the inverse function theorem. This result is classic; see [SY77, pp. 247–248], [GW79, pp. 160–161], and [TY90, p. 582].

A subtlety is that the radius r could depend on P a priori. Indeed, the complex structure on $B(0;R)$ is pulled back from the complex manifold by $\exp_P$. Consequently, for different P, the corresponding $B(0;R)$ is in general different as a complex manifold. Thus, the $\bar\partial$-operator is different on different balls; so is the radius r obtained from the L^2 estimate of $\bar\partial$ and the inverse function theorem.

We remark that the subtlety is not addressed in the classical works, as they do not need to. Either the Riemannian version of quasi-bounded geometry, or the explicit construction on quasi-projective manifolds, is sufficient for applications in [TY90].

The subtlety is settled in Lemma 12, an indispensable ingredient in the proof of Theorem 9, which is, in turn, crucial for proving the Greene-Wu conjectures. We first reformulate the classical result into the form of the following Lemma 10, from which we can proceed further. In the proof of Lemma 12, we transform the $\bar\partial$-equation into the Laplace equation for functions (in contrast to those [FK72, (1.1.1)] for $(0,1)$-forms); this allows us to apply maximum principles. In this approach, the constants of estimates depend only on the curvature bounds; so does the radius.

Lemma 10. *Let (N^n, g) be an n-dimensional Kähler manifold, and let $B(P;\delta_0)$ be an open geodesic ball of radius δ_0 centered at a point P in N. Suppose that $B(P;\delta_0)$ is contained in a coordinate chart in N with smooth, real-valued, coordinate functions $\{x^1,\dots,x^n,x^{n+1},\dots,x^{2n}\}$. Assume that the following conditions hold, where each A_j denotes a positive constant.*

(i) *No cut point of P is contained in $B(P;\delta_0)$.*
(ii) *The sectional curvature $K(g)$ of g satisfies $-A_2 \le K(g) \le A_1$ on $B(P;\delta_0)$.*
(iii) *For each $j = 1,\dots,n$,*
$$|\bar\partial(x^j + \sqrt{-1}\,x^{n+j})|_g(Q) \le \phi(r), \quad \text{for all } Q \in B(P;\delta_0).$$
Here $r = r(Q)$ denotes the geodesic distance $d(P,Q)$, and $\phi \ge 0$ is a continuous function on $[0,+\infty)$ satisfying
$$\int_0^{1/2} \frac{\phi^2(t)}{t^3}\,dt < +\infty. \tag{2.4}$$

Then, there exists a system of holomorphic coordinates $\{v^1,\dots,v^n\}$ defined on a smaller geodesic ball $B(P;\delta_1)$ such that
$$v^j = x^j + \sqrt{-1}\,x^{n+j}, \quad dv^j = d(x^j + \sqrt{-1}\,x^{n+j}), \quad \text{at } P \tag{2.5}$$
for all $j = 1,\dots,n$.

Proof. Let h be a real-valued smooth function on $[0, +\infty)$ and let ω_g be the Kähler form of g. By conditions (i) and (ii), we apply the Hessian Comparison Theorem (see, for example, [SY77, p. 231] and [SY94, p. 4, Theorem 1.1]) to $h(r)$ to obtain

$$4dd^c h(r) \ge \min\left\{2h'(r)\sqrt{A_1}\cot(\sqrt{A_1}\,r), h'(r)\sqrt{A_1}\cot(\sqrt{A_1}\,r) + h''(r)\right\}\omega_g,$$

for all $x \in B(P; \delta_0)$, where $r = r(x) = d(x, P)$. Letting $h(r)$ be r^2 and $\log(1 + r^2)$, respectively, yields

$$dd^c r^2 \ge \frac{\pi}{4}\omega_g, \tag{2.6}$$
$$dd^c \log(1 + r^2) \ge \frac{4\pi}{17}\omega_g,$$

for all $x \in B(P; \delta)$, where δ is a constant satisfying

$$0 < \delta \le \min\left\{\delta_0, \frac{1}{4}, \frac{\pi}{4\sqrt{A_1}}\right\}. \tag{2.7}$$

Inequality (2.6) in particular implies that $B(P; \delta)$ is a Stein manifold. On the other hand, by (ii), the Ricci curvature $\mathrm{Ric}(\omega_g)$ of g satisfies

$$|\mathrm{Ric}(\omega_g)|_g \le \sqrt{n}|R_{\mathrm{m}}|_g \le \frac{34}{3}n^{3/2}(A_1 + A_2).$$

Pick a constant $l > 0$ such that

$$\frac{4\pi}{17}l \ge \frac{34}{3}n^{3/2}(A_1 + A_2) + 1,$$

and let

$$\varphi_1 = l\log(1 + r^2), \quad \varphi_2 = (2n + 2)\log r, \quad \varphi = \varphi_1 + \varphi_2.$$

Then,

$$dd^c\varphi_1 + \mathrm{Ric}(\omega_g) \ge \omega_g \quad \text{on } B(P; \delta).$$

Let $0 \le \chi \le 1$ be a smooth function on $\mathbb{R}$ such that $\chi \equiv 1$ on the closed interval $[0, \delta/6]$ and $\chi \equiv 0$ on $[\delta/3, +\infty)$. Let

$$w^j = x^j + \sqrt{-1}x^{n+j}, \quad j = 1, \ldots, n.$$

It follows from [SY77, Proposition 2.1, p. 244–245] (see also [MSY81, Lemma 4 and Remark, p. 208] for Stein Kähler manifolds) that there is a smooth function β^j on $B(P; \delta)$ such that

$$\bar\partial\beta^j = \bar\partial[(\chi \circ r)w^j] \quad \text{on } B(P; \delta) \tag{2.8}$$

and satisfies

$$\int_{B(P;\delta)} |\beta^j|^2 e^{-\varphi} dV_g \le \int_{B(P;\delta)} |\bar\partial((\chi \circ r)w^j)|_g^2 e^{-\varphi} dV_g, \quad j = 1\ldots, n. \tag{2.9}$$

By condition (iii),

$$\int_{B(P;\delta/6)} |\bar\partial((\chi\circ r)w^j)|_g^2 e^{-\varphi} dV_g = \int_{B(P;\delta/6)} |\bar\partial w^j|_g^2 e^{-\varphi} dV_g \le C(n,A_1,A_2)\int_0^{\delta/6} \frac{\phi^2(r)}{r^3}dr \le C(n,A_1,A_2) < +\infty, \tag{2.10}$$

where we use the standard volume comparison $dV_g \le C(n,A_1,A_2)r^{2n-1}dr dV_{\mathbb{S}^{2n-1}}$ for $r\le\delta\le 1/4$, and $C(n,A_1,A_2)>0$ denotes a generic constant depending only on n, A_1, A_2. This together with (2.9) imply

$$\beta^j = 0, \quad d\beta^j = 0 \quad \text{at } P.$$

Let

$$v^j = (\chi\circ r)w^j - \beta^j, \quad j = 1,\ldots,n. \tag{2.11}$$

Then v^j is holomorphic and satisfies (2.5) for each j. By the inverse function theorem, the set of functions $\{v^1,\ldots,v^n\}$ forms a holomorphic coordinate system in a smaller ball $B(P;\delta_1)$ where $0<\delta_1<\frac{1}{6}\min\{\,\delta_0, 1/4, \pi/(4\sqrt{A_1}\,)\}$. □

Remark 11. Condition (2.5) in particular includes two cases, $\phi(t) = t^{1+a}$ with constant $a>0$, and $\phi(t) = t^k(-\log t)^{-l}$ with $k,l\ge 1$. The former is sufficient for our current application. For clarity, we specify $\phi(t)=t^{1+a}$ in the lemma below.

Lemma 12. *Let (N^n, g) and $B(P;\delta_0)$ be given as in Lemma 10, satisfying conditions (i), (ii), and (iii) with $\phi(r) = A_3 r^{1+\sigma}$ for some constant $\sigma>0$. Assume, in addition, that the metric component $\{g_{ij}\}$ of g with respect to $\{x^1,\ldots,x^{2n}\}$ satisfies*

$$A_4^{-1}(\delta_{ij}) \le (g_{ij})(Q) \le A_4(\delta_{ij}), \qquad 1\le i,j\le 2n, \tag{2.12}$$

$$\Big|\frac{\partial g_{ij}}{\partial x^k}(Q)\Big| \le A_5, \qquad 1\le i,j,k\le 2n, \tag{2.13}$$

for all $Q\in B(P;\delta_0)$. Then, there is a holomorphic coordinate system $\{v^1,\ldots,v^n\}$ defined on a smaller geodesic ball $B(P;\delta_1)$, for which

(a) *the radius δ_1 depends only on δ_0, n, A_j, $1\le j\le 5$, and also σ if $\sigma<1$;*
(b) *the coordinate function v^j satisfies (2.5) and*

$$|v^j - w^j| \le \frac{1}{2}r^{1+\frac{\sigma_1}{2}}, \tag{2.14}$$

$$\Big|\frac{\partial v^i}{\partial w^j} - \delta_{ij}\Big| \le \frac{1}{2}r^{\frac{\sigma_1}{2}}, \quad \Big|\frac{\partial v^i}{\partial \overline{w}^j}\Big| \le \frac{1}{2}r^{\frac{\sigma_1}{2}}, \tag{2.15}$$

on $B(P;\delta_1)$ for all $1\le i,j\le n$, where $w^j = x^j + \sqrt{-1}x^{n+j}$, $r = r(Q) = d(P,Q)$, and $\sigma_1 \equiv \min\{\sigma, 1\}$.

Proof. It remains to show (a) and (2.14). We start from (2.8) to obtain

$$\bar\partial^*\bar\partial\beta^j = \bar\partial^*\bar\partial[(\chi\circ r)w^j] \quad \text{on } B(P;\delta).$$

where $\delta > 0$ is a constant satisfying (2.7) and $w^j = x^j + \sqrt{-1}x^{n+j}$. Since (N, g) is Kähler, the Laplace-Beltrami operator Δ_g is equal to the $\bar\partial$-Laplacian $\Box = \bar\partial^*\bar\partial + \bar\partial\bar\partial^*$ up to a constant factor (-2) (i.e., $\Delta_g = -2\Box$). It follows that

$$\Delta_g \beta^j = \Delta_g w^j \equiv f \quad \text{on } B(P; \delta/6). \tag{2.16}$$

One can write

$$\Delta_g = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^a}\Big(g^{ab}\sqrt{g}\frac{\partial}{\partial x^b}\Big)$$

on $B(0; \delta)$ using the given single coordinate system $\{x^1, \ldots, x^{2n}\}$, where the summation notation is used and $1 \le a, b \le 2n$. It follows that Δ_g is of the divergence form, and is uniformly elliptic by (2.12).

Applying the standard interior estimate [GT01, p. 210, Theorem 8.32] to equation (2.16) yields

$$|d\beta^j|_{0;B(P;\delta/24)} \le C(n, A)\Big[\delta^{-1}|\beta^j|_{0;B(P;\delta/12)} + \delta|f|_{0;B(P;\delta/12)}\Big].$$

Here $|\cdot|_{0;U} \equiv |\cdot|_{C^0(U)}$ for a domain U, and we denote by $C(n, A)$ a generic constant depending only on n and A_j, $1 \le j \le 5$.

To estimate the C^0-norm, we use the local maximum principle ([GT01, Theorem 8.17, p. 194] with $\nu = 0$) to get

$$|\beta^j|_{0;B(P;\delta/12)} \le C(n, A)\Big[\delta^{-n}|\beta^j|_{L^2(B(P;\delta/6))} + \delta^2|f|_{0;B(P;\delta/6)}\Big]. \tag{2.17}$$

Combining these two estimates yields

$$|d\beta^j|_{0;B(P;\delta/24)} \le C(n, A)\Big[\delta^{-n-1}|\beta^j|_{L^2(B(P;\delta/6))} + \delta|f|_{0;B(P;\delta/6)}\Big].$$

To estimate the L^2-norm, we apply (2.9) and (2.10) to obtain

$$\begin{aligned}
|\beta^j|^2_{L^2(B(P;\delta/6))} &= \int_{B(P;\delta/6)} |\beta^j|^2 e^{-\varphi} e^{\varphi} dV_g \\
&\le \delta^{2n+2} \int_{B(P;\delta/6)} \big|\bar\partial((\chi\circ r)w^j)\big|^2_g e^{-\varphi} dV_g \\
&\le C(n, A_1, A_2, A_3)\delta^{2n+2}\int_0^{\delta/6} r^{2\sigma-1} dr \\
&\le C(n, A_1, A_2, A_3)\sigma^{-1}\delta^{2n+2+2\sigma}.
\end{aligned}$$

On the other hand, it follows from (2.12) and (2.13) that

$$|f|_{0;B(P;\delta/6)} \le C(n, A_4, A_5).$$

Hence,

$$\begin{aligned}
|d\beta^j|_{0;B(P;\delta/24)} &\le C(n, A)[\sigma^{-1}\delta^\sigma + \delta] \\
&\le C(n, A)\sigma_1^{-1}\delta^{\sigma_1}, \quad \sigma_1 \equiv \min\{1, \sigma\},
\end{aligned} \tag{2.18}$$

for all $1 \le j \le n$. It follows that

$$|\beta^j(Q)| \le C(n, A)\sigma_1^{-1} r^{1+\sigma_1} \quad \text{for any } Q \in B(P; \delta/24),$$

where $r = d(P, Q)$.

As in (2.11) we let

$$v^i = w^i - \beta^i \quad \text{on } B(P;\delta).$$

Then, for any $Q \in B(P;\delta/24)$,

$$\begin{aligned}|dv^1 \wedge \cdots \wedge dv^n|_g(Q) &\geq |dw^1 \wedge \cdots \wedge dw^n|_g(Q) - C(n,A)\sigma_1^{-1}\delta^{\sigma_1} \\ &\geq A_4^{-n/2} - C(n,A)\sigma_1^{-1}\delta^{\sigma_1},\end{aligned}$$

where we use (2.18) and (2.12). Moreover,

$$|v^j - w^j| = |\beta^j| \leq C(n,A)\sigma_1^{-1} r^{1+\sigma_1}.$$

Fix now a constant δ satisfying (2.7) and

$$C(n,A)\sigma_1^{-1}\delta^{\frac{\sigma_1}{2}} \leq \frac{A_4^{-n/2}}{2} \leq \frac{1}{2}.$$

Denote $\delta_1 = \delta/24$. It follows that $dv^1, \ldots, dv^n$ form an independent set at every point in $B(P;\delta_1)$; hence, $\{v^1, \ldots, v^n\}$ forms a coordinate system on $B(P;\delta_1)$ satisfying (2.14). Estimates (2.15) follows from $|d\beta^i(\partial/\partial w^k)| \leq |d\beta^i||\partial/\partial w^k|$, (2.12), and (2.18). □

Proof of Theorem 9. If (M, ω) has quasi-bounded geometry, then by definition the coordinate map ψ is a local biholomorphism. It then follows from (2.2) that the curvature R_{m} of ω and all its covariant derivatives are all bounded.

Conversely, if $|R_{\mathrm{m}}| \leq C_0$ then in particular the sectional curvature $K(\omega) \leq C_0$. It follows from the standard Rauch Comparison Theorem (see, for example, [dC92, p. 218, Proposition 2.4]) that for each $P \in M$, $B_\omega(P;R)$ contains no conjugate points of P for $R < \pi/\sqrt{C_0}$. Fix $R = \pi/(2\sqrt{C_0})$. Then, the exponential map

$$\exp_P : B(0;R) \subset T_{\mathbb{R},P}M \longrightarrow M \tag{2.19}$$

is nonsingular, and hence, a local diffeomorphism. The exponential map then pulls back a Kähler structure on $B(0;R)$ with Kähler metric $\exp_P^*\omega$ so that $\exp_P$ is a locally biholomorphic isometry. In particular, every geodesic in $B(0;R)$ through the origin is a straight line. Hence, $B(0;R)$ contains no cut point of the origin.

Pick an orthonormal basis $\{e_1, \ldots, e_{2n}\}$ of $T_{\mathbb{R},P}M$ with respect to $g \equiv \exp_P^*\omega$, such that the associated smooth coordinate functions $\{x^1, \ldots, x^{2n}\}$ on $T_{\mathbb{R},P}M$ satisfies

$$\bar\partial(x^j + \sqrt{-1}\,x^{n+j}) = 0 \quad \text{at } x = 0,$$

for each $j = 1, \ldots, n$. The complex-valued function $w^j \equiv x^j + \sqrt{-1}\,x^{n+j}$ need not be holomorphic. Nevertheless, we have the crucial Siu-Yau's inequality: If the sectional curvature $K(g)$ of g satisfies $-A_2 \leq K(g) \leq A_1$ with constant $A_1, A_2 > 0$, then

$$|\bar\partial w^j|_g \leq n^{5/2} A r^2 e^{A_2 r^2/6} \quad \text{on } B(0;R), \tag{2.20}$$

where

$$r = d(0,x) = |x| = \sqrt{(x^1)^2 + \cdots + (x^{2n})^2}, \quad x \in B(0;R),$$

and $A > 0$ is a constant depending only on A_1 and A_2. In fact, inequality (2.20) follows the same procedure of estimating the dual vector field as in [SY77, pp. 246–247] (its local version is also observed by [GW79, p. 159, (8.22)]), with two modifications given below, due to the different upper bounds for the sectional curvature. The inequality in [SY77, p. 246, line 11, i.e., p. 235, Proposition (1.5)] is replaced by

$$\Big|\nabla_{\partial/\partial r}\nabla_{\partial/\partial r}\Big(r\frac{\partial}{\partial x^l}\Big)\Big|_g \le n^2 Are^{A_2 r^2/6}, \quad 1 \le l \le 2n, \tag{2.21}$$

and the inequality $|X|^2 \ge \frac{1}{2}\sum_{j=1}^n(|\lambda_j|^2+|\mu_j|^2)$ in [SY77, p. 247] is replaced by

$$|X|^2 \ge \frac{1}{2}\min\Big\{1, \frac{\sin(\sqrt{A_1}r)}{\sqrt{A_1}r}\Big\}\sum_{j=1}^n(|\lambda_j|^2+|\mu_j|^2), \tag{2.22}$$

under the curvature condition $-A_2 \le K(g) \le A_1$; both (2.21) and (2.22) follow readily from the standard comparison argument ((2.22) is indeed half of (2.23) below).

Let $\{g_{ij}\}$ be the components of metric $g \equiv \exp_P^*\omega$ with respect to $\{x^j\}$. If the sectional curvature satisfies $-A_2 \le K(g) \le A_1$, then again by the standard Rauch comparison theorem we obtain

$$A^{-1}(\delta_{ij}) \le g_{ij}(x) \le A(\delta_{ij}), \quad 1 \le i,j \le 2n, \tag{2.23}$$

$$\Big|\frac{\partial g_{ij}}{\partial x^k}(x)\Big| \le n^4 Ar\exp\Big(\frac{A_2}{3}r^2\Big), \quad 1 \le i,j,k \le 2n, \tag{2.24}$$

for each $x \in B(0;R)$, where $r(x) = d(0,x)$ and $A > 0$ is a constant depending only on A_1 and A_2.

Thus, we can apply Lemma 12 with $B(P;\delta_0) = B(0;R)$ and $\phi(r) = C(n,C_0)r^2$ to obtain a smaller ball $B(0;\delta_1)$, on which there is a holomorphic coordinate system $\{v^1,\dots,v^n\}$ such that $v^j(0)=0$, $dv^j(0) = dw^j(0)$, $|v^j(x)-w^j(x)| \le \delta_1/(2\sqrt{n})$, and

$$\Big|\frac{\partial v^i}{\partial w^j}(x) - \delta_{ij}\Big| \le \frac{\delta_1}{2}, \quad \Big|\frac{\partial v^i}{\partial \overline{w}^j}(x)\Big| \le \frac{\delta_1}{2}, \tag{2.25}$$

for all $x \in B(0;\delta_1)$, $1 \le i,j \le n$. Here the radius $1/24 \ge \delta_1 > 0$ depends only on n and C_0. Since $\mathbf{v} \equiv (v^1,\dots,v^n)$ is biholomorphic from $B(0;\delta_1)$ onto its image U in $\mathbb{C}^n$, the image U satisfies

$$B_{\mathbb{C}^n}(0;\delta_1/2) \subset U \subset B_{\mathbb{C}^n}(0;3\delta_1/2).$$

It is now standard to verify that the composition $\exp_P \circ \mathbf{v}^{-1}$ is the desired quasi-coordinate map for P on U. Denote by $\{g_{i\bar{j}}\}$ the components of $\exp_P^*\omega$ with respect to coordinates $\{v^j\}$, by slightly abuse of notation. By (2.23) and (2.25), we obtain

$$C^{-1}(\delta_{ij}) \le (g_{i\bar{j}}) \le C(\delta_{ij}),$$

where $C > 0$ is a generic constant depending only on C_0 and n. This proves (2.1). The estimate of first order term $|\partial g_{i\bar{j}}/\partial v^k|$ follows from (2.24) and (2.25). The higher order estimate (2.2) follows from applying the standard Schauder estimate to the Ricci and scalar curvature equations [TY90, p. 582] (see also [DK81, p. 259, Theorem 6.1]).

For the second statement, fix a positive number $0 < R < r_\omega$, where r_ω denotes the injectivity radius of (M, ω). Then, for every $P \in M$, the exponential map given by (2.19), i.e., $\exp_P : B(0; R) \subset T_{\mathbb{R},P}M \to M$, is a diffeomorphism onto its image. From here the same process implies (M, ω) has bounded geometry. $\square$

We remark that the proof of Theorem 9 yields the following result: *A complete Kähler manifold (M, ω) of bounded curvature has quasi-bounded geometry of order zero, i.e., (M, ω) satisfies Definition 8 except (iii); furthermore, the radius of quasi-bounded geometry depends only on $\dim M$ and the curvature bounds. If in addition ω has positive injectivity radius, then (M, ω) has bounded geometry of order zero, and the radius of bounded geometry depends only on the curvature bounds, $\dim M$, and the injectivity radius of ω.* This result is sufficient for the sake of proving Conjectures 1 and 5. Our proof of Theorem 3 requires the full strength of Theorem 9, and hence, Lemma 13 in the next section.

3. Wan-Xiong Shi's Lemmas

The following lemma is useful to construct the quasi-bounded geometry.

Lemma 13. *Let (M, ω) be an n-dimensional complete noncompact Kähler manifold such that*

$$-\kappa_2 \leq H(\omega) \leq -\kappa_1 < 0 \tag{3.1}$$

for two constants $\kappa_1, \kappa_2 > 0$. Then, there exists another Kähler metric $\tilde{\omega}$ such that satisfying

$$C^{-1}\omega \leq \tilde{\omega} \leq C\omega, \tag{3.2}$$

$$-\tilde{\kappa}_2 \leq H(\tilde{\omega}) \leq -\tilde{\kappa}_1 < 0, \tag{3.3}$$

$$\sup_{x\in M} |\tilde{\nabla}^q \tilde{R}_{\alpha\bar{\beta}\gamma\bar{\sigma}}| \leq C_q, \tag{3.4}$$

where $\tilde{\nabla}^q \tilde{R}_{\mathrm{m}}$ denotes the qth covariant derivative of the curvature tensor $\tilde{R}_{\mathrm{m}}$ of $\tilde{\omega}$ with respect to $\tilde{\omega}$, and the positive constants $C = C(n)$, $\tilde{\kappa}_j = \tilde{\kappa}_j(n, \kappa_1, \kappa_2)$, $j = 1, 2$, $C_q = C_q(n, q, \kappa_1, \kappa_2)$ depend only on the parameters in their parentheses.

Lemma 13 (3.2) and (3.4) are contained in W. X. Shi [Shi97]. We provide below the details for the pinching estimate (3.3) of the holomorphic sectional curvature. Of course, if the manifold were compact, then (3.3) would follow trivially from the usual uniform continuity of a continuous function. However, this does not hold for a general bounded smooth function on a complete noncompact manifold. Here the maximum principle (Lemma 15 in Appendix A) has to be used.

In this section and Appendix A, we adopt the following convention: We denote by $\omega = (\sqrt{-1}/2)g_{\alpha\bar{\beta}}dz^\alpha \wedge d\bar{z}^\beta$ the Kähler form of a hermitian metric g_ω. The real part of the hermitian metric $g_\omega = g_{\alpha\bar{\beta}}dz^\alpha \otimes d\bar{z}^\beta$ induces a Riemannian metric $g = g_{ij}dx^i \otimes dx^j$ on $T_{\mathbb{R}}M$ which is compatible with the complex structure J. Extend g linearly over $\mathbb{C}$ to $T_{\mathbb{R}}M \otimes_{\mathbb{R}} \mathbb{C} = T'M \oplus \overline{T'M}$, and then restricting it to $T'M$ recovers $(1/2)g_\omega$; that is,

$$g(v, w) = \mathrm{Re}(g_\omega(\eta, \xi)), \quad g_\omega(\eta, \xi) = 2g(\eta, \bar{\xi}).$$

Here v, w are real tangent vectors, and η, ξ are their corresponding holomorphic tangent vectors under the $\mathbb{R}$-linear isomorphism $T_{\mathbb{R}}M \to T'M$, i.e., $\eta = \frac{1}{2}(v - \sqrt{-1}Jv)$, $\xi = \frac{1}{2}(w - \sqrt{-1}Jw)$. Then, the curvature tensor R_{m} satisfies

$$R(\eta, \overline{\eta}, \xi, \overline{\xi}) = \frac{1}{2}R(v, Jv, Jw, w).$$

It follows that

$$H(x, \eta) = R(\eta, \overline{\eta}, \eta, \overline{\eta}) = \frac{1}{2}R(v, Jv, Jv, v).$$

Unless otherwise indicated, the Greek letters such as α, β are used denote the holomorphic vectors $\partial/\partial z^\alpha, \partial/\partial z^\beta$ and range over $\{1, \ldots, n\}$, while the latin indices such as i, j, k are used to denote real vectors $\partial/\partial x^i, \partial/\partial x^j$ and range over $\{1, \ldots, 2n\}$.

Proof of Lemma 13. The assumption (3.1) on H implies the the curvature tensor R_{m} is bounded; more precisely,

$$\sup_{x\in M} |R_{\mathrm{m}}(x)| \le \frac{34}{3} n^2(\kappa_2 - \kappa_1).$$

Here and in many places of the proof, the constant in an estimate is given in certain explicit form, mainly to indicate its dependence on the parameters such as κ_i and n. Applying [Shi97, p. 99, Corollary 2.2] yields that the equation

$$\begin{cases} \dfrac{\partial}{\partial t} g_{ij}(x,t) = -2R_{ij}(x,t) \\ \quad g_{ij}(0,t) = g_{ij}(x) \end{cases}$$

admits a smooth solution $\{g_{ij}(x,t)\} > 0$ for $0 \le t \le \theta_0(n)/(\kappa_2 - \kappa_1)$, where $\theta_0(n) > 0$ is a constant depending only on n. Furthermore, the curvature $R_{\mathrm{m}}(x,t) = \{R_{ijkl}(x,t)\}$ of $\{g_{ij}(x,t)\}$ satisfies that, for each nonnegative integer q,

$$\sup_{x\in M} |\nabla^q R_{\mathrm{m}}(x,t)|^2 \le \frac{C(q,n)(\kappa_2-\kappa_1)^2}{t^q}, \quad \text{for all } 0 < t \le \frac{\theta_0(n)}{\kappa_2 - \kappa_1} \equiv T, \tag{3.5}$$

where $C(q,n) > 0$ is a constant depending only q and n. In particular, the metric $g_{ij}(x,t)$ satisfies Assumption A in [Shi97, p. 120]. Then, by [Shi97, p. 129, Theorem 5.1], the metric $g_{ij}(x,t)$ is Kähler, and satisfies

$$\begin{cases} \dfrac{\partial}{\partial t} g_{\alpha\bar\beta}(x,t) = -4R_{\alpha\bar\beta}(x,t) \\ \quad g_{\alpha\bar\beta}(x,0) = g_{\alpha\bar\beta}(x), \end{cases}$$

for all $0 \le t \le T$. It follows that

$$e^{-tC(n)(\kappa_2-\kappa_1)} g_{\alpha\bar\beta}(x) \le g_{\alpha\bar\beta}(x,t) \le e^{tC(n)(\kappa_2-\kappa_1)} g_{\alpha\bar\beta}(x), \tag{3.6}$$

for all $0 \le t \le T = \theta_0(n)/(\kappa_2 - \kappa_1)$. Here and below, we denote by $C(n)$ and $C_j(n)$ generic positive constants depending only on n. Then, for an arbitrary $0 < t \le T$, the metric $\omega(x,t) = (\sqrt{-1}/2) g_{\alpha\bar\beta}(x,t) dz^\alpha \wedge d\bar z^\beta$ satisfies (3.2) and (3.4); in particular, the constant C in (3.2) depends only on n, since $tC(n)(\kappa_2 - \kappa_1) \le \theta_0(n)C(n)$.

We next to show that there exists a small $0 < t_0 \le T$ so that $\omega(x,t)$ also satisfies (3.3) whenever $0 < t \le t_0$. Recall that the curvature tensor satisfies the evolution equation (see, for example, [Shi97, p. 143, (122)])

$$\begin{aligned}\frac{\partial}{\partial t}R_{\alpha\bar\beta\gamma\bar\sigma} = {}& 4\Delta R_{\alpha\bar\beta\gamma\bar\sigma} + 4g^{\mu\bar\nu}g^{\rho\bar\tau}(R_{\alpha\bar\beta\mu\bar\tau}R_{\gamma\bar\sigma\rho\bar\nu} + R_{\alpha\bar\sigma\mu\bar\tau}R_{\gamma\bar\beta\rho\bar\nu} - R_{\alpha\bar\nu\gamma\bar\tau}R_{\mu\bar\beta\rho\bar\sigma})\\ &- 2g^{\mu\bar\nu}(R_{\alpha\bar\nu}R_{\mu\bar\beta\rho\bar\tau} + R_{\mu\bar\beta}R_{\alpha\bar\nu\rho\bar\tau} + R_{\gamma\bar\nu}R_{\alpha\bar\beta\mu\bar\sigma} + R_{\mu\bar\sigma}R_{\alpha\bar\beta\rho\bar\nu}),\end{aligned}$$

where $\Delta \equiv \Delta_{\omega(x,t)} = \frac{1}{2}g^{\alpha\bar\beta}(x,t)(\nabla_{\bar\beta}\nabla_\alpha + \nabla_\alpha\nabla_{\bar\beta})$. It follows that

$$\begin{aligned}&\Big(\frac{\partial}{\partial t}R_{\alpha\bar\beta\gamma\bar\sigma}\Big)\eta^\alpha\bar\eta^\beta\eta^\gamma\bar\eta^\sigma\\ &\le 4\Big(\Delta R_{\alpha\bar\beta\gamma\bar\sigma}\Big)\eta^\alpha\bar\eta^\beta\eta^\gamma\bar\eta^\sigma + C_1(n)|\eta|^4_{g_{\alpha\bar\beta}(x,t)}|R_{\mathrm{m}}(x,t)|^2_{\omega(x,t)}\\ &\le 4\Big(\Delta R_{\alpha\bar\beta\gamma\bar\sigma}\Big)\eta^\alpha\bar\eta^\beta\eta^\gamma\bar\eta^\sigma + C_1(n)(\kappa_2-\kappa_1)^2|\eta|^4_{\omega(x,t)}.\end{aligned} \tag{3.7}$$

by (3.5) with $q = 0$. Let

$$H(x,\eta,t) = \frac{R_{\alpha\bar\beta\gamma\bar\sigma}\eta^\alpha\bar\eta^\beta\eta^\gamma\bar\eta^\sigma}{|\eta|^4_{\omega(x,t)}}.$$

Then, by (3.1) and (3.5),

$$\begin{aligned}H(x,\eta,0) &\le -\kappa_1,\\ |H(x,\eta,t)| &\le |R_{\mathrm{m}}(x,t)|_{\omega(x,t)} \le C_0(n)(\kappa_2-\kappa_1).\end{aligned}$$

To apply the maximum principle (Lemma 15 in Appendix A), we denote

$$h(x,t) = \max\{H(x,\eta,t); |\eta|_{\omega(x,t)} = 1\},$$

for all $x \in M$ and $0 \le t \le \theta_0(n)/(\kappa_2-\kappa_1)$. Then, h with (3.7) satisfy the three conditions in Lemma 15. It follows that

$$h(x,t) \le C_2(n)(\kappa_2-\kappa_1)^2 t - \kappa_1.$$

where $C_2(n) = C_1(n) + 8\sqrt{n}C_0(n)^2 > 0$. Let

$$t_0 = \min\Big\{\frac{\kappa_1}{2C_2(n)(\kappa_2-\kappa_1)^2}, \frac{\theta_0(n)}{\kappa_2-\kappa_1}\Big\} > 0.$$

Then, for all $0 < t \le t_0$,

$$H(x,\eta,t) \le h(x,t) \le -\frac{\kappa_1}{2} < 0.$$

Since the curvature tensor is bounded (by (3.5) with $q = 0$), we have

$$H(x,\eta,t) \ge -C_0(n)(\kappa_2-\kappa_1).$$

Thus, for an arbitrary $t \in (0,t_0]$, the metric $\omega(x,t) = (\sqrt{-1}/2)g_{\alpha\bar\beta}(x,t)dz^\alpha \wedge d\bar z^\beta$ is a desired metric satisfying (3.3), and also (3.2) and (3.4). $\square$

Lemma 14. *Let (M^n, ω) be a complete noncompact Kähler manifold whose Riemannian sectional curvature is pinched between two negative constants, i.e.,*

$$-\kappa_2 \le K(\omega) \le -\kappa_1 < 0.$$

Then, there exists another Kähler metric $\tilde{\omega}$ satisfying

$$\begin{aligned} &C^{-1}\omega \le \tilde{\omega} \le C\omega, \\ &-\tilde{\kappa}_2 \le K(\tilde{\omega}) \le -\tilde{\kappa}_1 < 0, \\ &\sup_{x\in M} |\tilde{\nabla}^q \tilde{R}_{\alpha\bar{\beta}\gamma\bar{\sigma}}| \le C_q, \end{aligned}$$

where $\tilde{\nabla}^q \tilde{R}_{\alpha\bar{\beta}\gamma\bar{\sigma}}$ denotes the qth covariant derivatives of $\{\tilde{R}_{\alpha\bar{\beta}\gamma\bar{\sigma}}\}$ with respect to $\tilde{\omega}$, and the positive constants $C = C(n)$, $\tilde{\kappa}_j = \tilde{\kappa}_j(n, \kappa_1, \kappa_2)$, $j = 1, 2$, $C_q = C_q(n, q, \kappa_1, \kappa_2)$ depend only on the parameters inside their parentheses.

The proof of Lemma 14 is entirely similar to that of Lemma 13, with the following modification: The function φ is now given by

$$\varphi(x, v, w, t) = R_{ijkl}(x, t)v^i w^j w^k v^l,$$

for any $x \in M$ and $v, w \in T_{\mathbb{R},x}M$, and

$$\begin{aligned} h(x,t) &= \max\{\varphi(x,\eta,\xi,t); |\eta\wedge\xi|_{g(x,t)} = 1\} \\ &= \max\{\varphi(x,\eta,\xi,t); |\eta|_{g(x,t)} = |\xi|_{g(x,t)} = 1, \langle\eta,\xi\rangle_{g(x,t)} = 0\}. \end{aligned}$$

Here $|\eta\wedge\xi|^2 = |\eta|^2|\xi|^2 - \langle\eta,\xi\rangle^2$. The result then follows from Lemma 16.

A. Maximum principles

The proof of Lemma 13 uses the following maximum principle, which extends [Shi97, p. 124, Lemma 4.7] to tensors; compare [Shi97, pp. 145–147], [Ham82, Theorem 9.1], and [CCCY03, pp. 139–140], for example.

Let $(M, \tilde{\omega})$ be an n-dimensional complete noncompact Kähler manifold. Suppose for some constant $T > 0$ there is a smooth solution $\omega(x,t) > 0$ for the evolution equation

$$\begin{cases} \dfrac{\partial}{\partial t} g_{\alpha\bar{\beta}}(x,t) = -4R_{\alpha\bar{\beta}}(x,t), & \text{on } M\times[0,T], \\ g_{\alpha\bar{\beta}}(x,0) = \tilde{g}_{\alpha\bar{\beta}}(x), & x \in M, \end{cases} \tag{A.1}$$

where $g_{\alpha\bar{\beta}}(x,t)$ and $\tilde{g}_{\alpha\bar{\beta}}$ are the metric components of $\omega(x,t)$ and $\tilde{\omega}$, respectively. Assume that the curvature $R_{\mathrm{m}}(x,t) = \{R_{\alpha\bar{\beta}\gamma\bar{\sigma}}(x,t)\}$ of $\omega(x,t)$ satisfies

$$\sup_{M\times[0,T]} |R_{\mathrm{m}}(x,t)|^2 \le k_0 \tag{A.2}$$

for some constant $k_0 > 0$.

Lemma 15. *With the above assumption, suppose a smooth tensor* $\{W_{\alpha\bar{\beta}\gamma\bar{\sigma}}(x,t)\}$ *on* M *with complex conjugation* $\overline{W}_{\alpha\bar{\beta}\gamma\bar{\sigma}}(x,t) = W_{\beta\bar{\alpha}\sigma\bar{\gamma}}(x,t)$ *satisfies*

$$\Big(\frac{\partial}{\partial t} W_{\alpha\bar{\beta}\gamma\bar{\sigma}}\Big)\eta^{\alpha}\bar{\eta}^{\beta}\eta^{\gamma}\bar{\eta}^{\sigma} \le \big(\Delta W_{\alpha\bar{\beta}\gamma\bar{\sigma}}\big)\eta^{\alpha}\bar{\eta}^{\beta}\eta^{\gamma}\bar{\eta}^{\sigma} + C_1|\eta|^4_{\omega(x,t)}, \tag{A.3}$$

for all $x \in M$, $\eta \in T'_x M$, $0 \le t \le T$, *where* $\Delta \equiv 2g^{\alpha\bar{\beta}}(x,t)(\nabla_{\bar{\beta}}\nabla_{\alpha} + \nabla_{\alpha}\nabla_{\bar{\beta}})$ *and* C_1 *is a constant. Let*

$$h(x,t) = \max\Big\{W_{\alpha\bar{\beta}\gamma\bar{\sigma}}\eta^{\alpha}\bar{\eta}^{\beta}\eta^{\gamma}\bar{\eta}^{\sigma}; \eta \in T'_x M, |\eta|_{\omega(x,t)} = 1\Big\},$$

for all $x \in M$ *and* $0 \le t \le T$. *Suppose*

$$\sup_{x\in M, 0\le t\le T} |h(x,t)| \le C_0, \tag{A.4}$$

$$\sup_{x\in M} h(x,0) \le -\kappa, \tag{A.5}$$

for some constants $C_0 > 0$ *and* κ. *Then,*

$$h(x,t) \le (8C_0\sqrt{nk_0} + C_1)t - \kappa.$$

for all $x \in M$, $0 \le t \le T$.

Proof. We prove by contradiction. Denote

$$C = 8C_0\sqrt{nk_0} + C_1 > 0. \tag{A.6}$$

Suppose

$$h(x_1,t_1) - Ct_1 + \kappa > 0 \tag{A.7}$$

for some $(x_1,t_1) \in M \times [0,T]$. Then, by (A.5) we have $t_1 > 0$.

Under the above conditions (A.1) and (A.2), by [Shi97, p. 124, Lemma 4.6], there exists a function $\theta(x,t) \in C^{\infty}(M \times [0,T])$ satisfying that

$$0 < \theta(x,t) \le 1, \qquad \text{on } M \times [0,T], \tag{A.8}$$

$$\frac{\partial\theta}{\partial t} - \Delta_{\omega(x,t)}\theta + 2\theta^{-1}|\nabla\theta|^2_{\omega(x,t)} \le -\theta, \qquad \text{on } M \times [0,T], \tag{A.9}$$

$$\frac{C_2^{-1}}{1 + d_0(x_0,x)} \le \theta(x,t) \le \frac{C_2}{1 + d_0(x_0,x)}, \qquad \text{on } M \times [0,T]. \tag{A.10}$$

where x_0 is a fixed point in M, $d_0(x,y)$ is the geodesic distance between x and y with respect to $\omega(x,0)$, and $C_2 > 0$ is a constant depends only on n, k_0, and T.

Let

$$m_0 = \sup_{x\in M, 0\le t\le T}\Big(\big[h(x,t) - Ct + \kappa\big]\theta(x,t)\Big).$$

Then, $0 < m_0 \le C_0 + |\kappa|$, by (A.7) and (A.8). Denote

$$\Lambda = \frac{2C_2(C_0 + CT + |\kappa|)}{m_0} > 0.$$

Then, for any $x \in M$ with $d_0(x, x_0) \geq \Lambda$,

$$\Big|(h(x,t) - Ct + \kappa)\theta(x,t)\Big| \leq \frac{C_2(C_0 + CT + |\kappa|)}{1 + d_0(x, x_0)} \leq \frac{m_0}{2}.$$

It follows that the function $(h - Ct + \kappa)\theta$ must attain its supremum m_0 on the compact set $\overline{B(x_0;\Lambda)} \times [0,T]$, where $\overline{B(x_0;r)}$ denotes the closure of the geodesic ball with respect to $\omega(x,0)$ centered at x_0 of radius r. Let

$$f(x,\eta,t) = \frac{W_{\alpha\bar\beta\gamma\bar\sigma}\eta^\alpha\bar\eta^\beta\eta^\gamma\bar\eta^\sigma}{|\eta|^4_{\omega(x,t)}} - Ct + \kappa,$$

for all $(x,t) \in M \times [0,T]$, $\eta \in T'_x M \setminus \{0\}$. Then, there exists a point (x_*, η_*, t_*) with $x_* \in \overline{B(x_0;\Lambda)}$, $0 \leq t_* \leq T$, $\eta_* \in T'_{x_*}M$ and $|\eta_*|_{\omega(x_*,t_*)} = 1$, such that

$$m_0 = f(x_*, \eta_*, t_*)\theta(x_*, t_*) = \max_{\mathcal{S}_t \times [0,T]} (f\theta),$$

and $t_* > 0$ by (A.5), where $\mathcal{S}_t = \{(x,\eta) \in T'M; x \in M, \eta \in T'_x M, |\eta|_{\omega(x,t)} = 1\}$.

We now employ a standard process to extend η_* to a smooth vector field, denoted by η with slightly abuse of notation, in a neighborhood of (x_*, t_*) in $M \times [0,T]$ such that η is nowhere vanishing on the neighborhood, and

$$\frac{\partial}{\partial t}\eta = 0, \quad \nabla\eta = 0, \quad \Delta\eta = 0, \quad \text{at } (x_*, t_*). \tag{A.11}$$

This extension can be done, for example, by parallel transporting η_* from x_* to each point y in a small geodesic ball centered at x_*, with respect to metric $\omega(\cdot, t_*)$, along the unique minimal geodesic joining x_* to y; this extension is made independent of t and so $\partial\eta/\partial t \equiv 0$ in the geodesic ball.

Since $f(x, \eta(x), t)$ is smooth in a neighborhood of (x_*, t_*), we can differentiate f and evaluate the derivatives at the point (x_*, t_*) to obtain

$$\begin{aligned}
\frac{\partial}{\partial t} f &= \Big(\frac{\partial}{\partial t} W_{\alpha\bar\beta\gamma\bar\sigma}\Big)\eta^\alpha\bar\eta^\beta\eta^\gamma\bar\eta^\sigma + 8\Big(W_{\alpha\bar\beta\gamma\bar\sigma}\eta^\alpha\bar\eta^\beta\eta^\gamma\bar\eta^\sigma\Big)\Big(R_{\alpha\bar\beta}\eta^\alpha\bar\eta^\beta\Big) - C \\
&\leq \Big(\frac{\partial}{\partial t} W_{\alpha\bar\beta\gamma\bar\sigma}\Big)\eta^\alpha\bar\eta^\beta\eta^\gamma\bar\eta^\sigma + 8C_0\sqrt{nk_0} - C \quad \big(\text{by (A.4) and (A.2)}\big) \\
&\leq \Delta f + C_1 + 8C_0\sqrt{nk_0} - C \quad \big(\text{by (A.3) and (A.11)}\big) \\
&\leq \Delta f, \quad \text{by (A.6)}.
\end{aligned}$$

Since $f\theta = f(x, \eta(x), t)\theta(x,t)$ attains its maximum at (x_*, t_*), we have

$$\frac{\partial}{\partial t}(f\theta) \geq 0, \quad \nabla(f\theta) = 0, \quad \Delta(f\theta) \leq 0, \quad \text{at } (x_*, t_*). \tag{A.12}$$

It follows that, at the point (x_*, t_*),

$$\begin{aligned}
0 \le \frac{\partial}{\partial t}(f\theta) &= \theta\frac{\partial}{\partial t}f + f\frac{\partial}{\partial t}\theta \\
&\le \theta\Delta f + f\frac{\partial}{\partial t}\theta \\
&= \Delta(f\theta) - 2\theta^{-1}\nabla\theta\cdot\nabla(f\theta) + f\Big[\frac{\partial\theta}{\partial t} - \Delta\theta + 2\theta^{-1}|\nabla\theta|^2\Big] \\
&\le -f\theta \quad \big(\text{by (A.12) and (A.9)}\big) \\
&= -m_0 < 0.
\end{aligned}$$

This yields a contradiction. The proof is therefore completed. □

In the proof of Lemma 13, we apply Lemma 15 with $W_{\alpha\bar\beta\gamma\bar\sigma} = R_{\alpha\bar\beta\gamma\bar\sigma}$ to estimate the holomorphic sectional curvature. For the Riemannian sectional curvature, we apply the similar result given below with $W_{ijkl} = R_{ijkl}$.

Lemma 16. *Assume (A.1) and (A.2). Suppose a smooth real tensor $\{W_{ijkl}(x,t)\}$ on M satisfies*

$$\Big(\frac{\partial}{\partial t}W_{ijkl}\Big)v^i w^j w^k v^l \le (\Delta W_{ijkl})v^i w^j w^k v^l + C_1|v|^2_{\omega(x,t)}|w|^2_{\omega(x,t)},$$

for all $x \in M$, $v, w \in T_{\mathbb{R},x}M$, where $\Delta \equiv g^{ij}(x,t)\nabla_i\nabla_j$ and $C_1 > 0$ is a constant. Let

$$k(x,t) = \max\Big\{W_{ijkl}v^i w^j w^k v^l; v, w \in T_{\mathbb{R},x}M, |v\wedge w|_{g(x,t)} = 1\Big\},$$

for all $x \in M$ and $0 \le t \le T$. Suppose

$$\sup_{x\in M, 0\le t\le T} |k(x,t)| \le C_0,$$
$$\sup_{x\in M} k(x,0) \le -\kappa,$$

for some constants $C_0 > 0$ and κ. Then,

$$k(x,t) \le (8C_0\sqrt{nk_0} + C_1)t - \kappa.$$

for all $x \in M$, $0 \le t \le T$. □

4. Kobayashi-Royden metric and holomorphic curvature

The *Kobayashi-Royden pseudometric*, denoted by $\mathfrak{K}$, is the infinitesimal form of the Kobayashi pseudodistance. Let us first recall the definition (see, for example, [Roy71] or [Kob98, Section 3.5]).

Let M be a complex manifold and $T'M$ be its holomorphic tangent bundle. Define $\mathfrak{K}_M : T'M \to [0, +\infty)$ as below: For any $(x,\xi) \in T'M$,

$$\mathfrak{K}_M(x,\xi) = \inf_{R>0} \frac{1}{R},$$

where R ranges over all positive numbers for which there is a $\phi \in \mathrm{Hol}(\mathbb{D}_R, M)$ with $\phi(0) = x$ and $\phi_*(\partial/\partial z|_{z=0}) \equiv d\phi(\partial/\partial z|_{z=0}) = \xi$. Here $\mathrm{Hol}(X, Y)$ denotes the set of holomorphic maps from X to Y, and

$$\mathbb{D}_R \equiv \{z \in \mathbb{C}; |z| < R\}, \quad \text{and } \mathbb{D} \equiv \mathbb{D}_1.$$

Equivalently, one can verify that (cf. [GW79, p. 82]), for each $(x, \xi) \in T'M$,

$$\begin{aligned} \mathfrak{K}_M(x,\xi) &= \inf\{|V|_{\mathcal{P}}; V \in T'\mathbb{D}, \text{ there is } f \in \mathrm{Hol}(\mathbb{D}, M) \text{ with } f_*(V) = \xi\} \\ &= \inf\{|V|_{\mathbb{C}}; V \in T'_0\mathbb{D}, \text{ there is } f \in \mathrm{Hol}(\mathbb{D}, M) \text{ such that} \\ &\qquad f(0) = x, f_*(V) = \xi\}. \end{aligned} \tag{4.1}$$

Here $|\cdot|_{\mathcal{P}}$ and $|\cdot|_{\mathbb{C}}$ are, respectively, the norms with respect to the Poincaré metric $\omega_{\mathcal{P}} = (\sqrt{-1}/2)(1-|z|^2)^{-2} dz \wedge d\bar{z}$ and Euclidean metric $\omega_{\mathbb{C}} = (\sqrt{-1}/2) dz \wedge d\bar{z}$.

The following decreasing property of $\mathfrak{K}_M$ follows immediately from definition.

Proposition 17 ([Roy71, Proposition 1])**.** *Let M and N be complex manifolds and $\Psi : M \to N$ be a holomorphic map. Then,*

$$(\Psi^*\mathfrak{K}_N)(x,\xi) \equiv \mathfrak{K}_N(\Psi(x), \Psi_*(\xi)) \le \mathfrak{K}_M(x,\xi)$$

for all $(x,\xi) \in T'M$. In particular, if $\Psi : M \to N$ is biholomorphism then the equality holds; if M is a complex submanifold of N then

$$\mathfrak{K}_N(x,\xi) \le \mathfrak{K}_M(x,\xi).$$

Example 18. Let M be the open ball $B(r) = \{z \in \mathbb{C}^n; |z| < r\}$. Then,

$$\mathfrak{K}_{B(r)}(a,\xi) = \left[\frac{|\xi|^2_{\mathbb{C}^n}}{r^2 - |a|^2} + \frac{|\xi \cdot a|^2_{\mathbb{C}^n}}{(r^2 - |a|^2)^2}\right]^{1/2}, \tag{4.2}$$

for all $a \in \mathbb{B}^n_r$ and $\xi \in T'_a\mathbb{B}^n_r$; see [JP13, p. 131, Example 3.5.6] for example. $\square$

The result below is well-known. We include a proof here for completeness.

Lemma 19. *Let (M, ω) be a hermitian manifold such that the holomorphic sectional curvature $H(\omega) \le -\kappa < 0$. Then,*

$$\mathfrak{K}_M(x,\xi) \ge \sqrt{\frac{\kappa}{2}}\, |\xi|_\omega \quad \text{for each } x \in M,\ \xi \in T'_xM.$$

Proof. Let $\psi \in \mathrm{Hol}(\mathbb{D}, M)$ such that $\psi(0) = x$ and $\psi_*(v) = \xi$. It follows from the second author's Schwarz Lemma [Yau78a, p. 201, Theorem 2′] that

$$\psi^*\omega \le \frac{2}{\kappa}\omega_{\mathcal{P}} \quad \text{on } \mathbb{D},$$

where $\omega_{\mathcal{P}} = (\sqrt{-1}/2)(1-|z|^2)^{-2} dz \wedge d\bar{z}$. It follows that

$$|\xi|^2_\omega = \omega(x;\xi) = (\psi^*\omega)(0;v) \le \frac{2}{\kappa}\omega_{\mathcal{P}}(0;v) = \frac{2}{\kappa}|v|^2_{\mathbb{C}}.$$

Hence, $|v|_{\mathbb{C}} \ge \sqrt{\kappa/2}\,|\xi|_\omega$. By (4.1), we obtain $\mathfrak{K}_M(x,\xi) \ge \sqrt{\kappa/2}\,|\xi|_\omega$. $\square$

The quasi-bounded geometry is essential in the following estimate.

Lemma 20. *Suppose a complete Kähler manifold (M,ω) has quasi-bounded geometry. Then, the Kobayashi-Royden pseudometric $\mathfrak{K}$ satisfies*

$$\mathfrak{K}_M(x,\xi) \le C|\xi|_\omega, \quad \text{for all } x \in M, \xi \in T'_x M,$$

where C depends only on the radius of quasi-bounded geometry of (M,ω).

Proof. Let $(\psi, B(r))$ be the quasi-coordinate chart of (M,ω) centered at x; that is, $B(r) = \{z \in \mathbb{C}^n; |z| < r\}$ and $\psi : B(r) \to M$ is nonsingular holomorphic map such that $\psi(0) = x$. Denote $U = \psi(B(r))$. Then, by Proposition 17,

$$\mathfrak{K}_M(x,\xi) \le \mathfrak{K}_U(x,\xi) = \mathfrak{K}_U(\psi(0), \psi_*(v)) \le \mathfrak{K}_{B(r)}(0,v),$$

where $v \in T'_0(B(r))$ such that $\phi_*(v) = \xi$. It follows from (4.2) that

$$\mathfrak{K}_M(x,\xi) \le \mathfrak{K}_{B(r)}(0,v) = \frac{|v|_{\mathbb{C}^n}}{r}.$$

By virtue of the quasi-bounded geometry of (M,ω), more precisely, (2.1), we have

$$C^{-1}|v|^2_{\mathbb{C}^n} \le (\psi^*\omega)(0;v) = \omega(x;\xi) \equiv |\xi|^2_\omega \le C|v|^2_{\mathbb{C}^n},$$

where $C > 0$ is a constant depending only on r. Hence,

$$\mathfrak{K}_M(x,\xi) \le \frac{\sqrt{C}}{r}|\xi|_\omega.$$

This completes the proof. ∎

Proof of Theorem 2. Since $-B \le H(\omega) \le -A$, we can assume (M,ω) has quasi-bounded geometry, by Lemma 13 and Theorem 9. Then, the radius of quasi-bounded geometry depends only on A, B, and $\dim M$. The desired result then follows from Lemma 20 and Lemma 19. ∎

5. Bergman metric and sectional curvature

Let M be an n-dimensional complex manifold. We follow [GW79, Section 8] for some notations. Let $\Lambda^{(n,0)}(M) \equiv A^{n,0}(M)$ be the space of smooth complex differential $(n,0)$ forms on M. For $\varphi, \psi \in \Lambda^{(n,0)}$, define

$$\langle \varphi, \psi \rangle = (-1)^{n^2/2} \int_M \varphi \wedge \overline{\psi} \tag{5.1}$$

and

$$\|\varphi\| = \sqrt{\langle \varphi, \varphi \rangle}.$$

Let $L^2_{(n,0)}$ be the completion of

$$\{\varphi \in \Lambda^{(n,0)}; \|\varphi\| < +\infty\}$$

with respect to $\|\cdot\|$. Then $L^2_{(n,0)}$ is a separable Hilbert space with the inner product $\langle \cdot, \cdot \rangle$. Define

$$\mathcal{H} = \{\varphi \in L^2_{(n,0)} \mid \varphi \text{ is holomorphic}\}.$$

Suppose $\mathcal{H} \neq \{0\}$. Let $\{e_j\}_{j\geq 0}$ be an orthonormal basis of $\mathcal{H}$ with respect to the inner product $\langle \cdot, \cdot \rangle$. Then, the $2n$ form defined on $M \times M$ given by

$$\mathfrak{B}(x,y) = \sum_{j\geq 0} e_j(x) \wedge \overline{e}_j(y), \qquad x, y \in M,$$

is the Bergman kernel of M. The convergence of this series is uniform on every compact subset of $M \times M$ (see also Lemma 21 below). The definition of $\mathfrak{B}(x,y)$ is independent of the choice of the orthonormal basis of $\mathcal{H}$. Let

$$\mathfrak{B}(x) = \mathfrak{B}(x,x) = \sum_{j\geq 0} e_j(x) \wedge \overline{e}_j(x) \qquad \text{for all } x \in M.$$

Then $\mathfrak{B}(x)$ is a smooth (n,n)-form on M, which is called the *Bergman kernel form* of M. Suppose for some point $P \in M$, $\mathfrak{B}(P) \neq 0$. Define

$$dd^c \log \mathfrak{B} = dd^c \log b$$

where we write $\mathfrak{B}(z) = b\, dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$ in terms of local coordinates $(z^1, \ldots, z^n)$. It is readily to check that this definition is well-defined. If $dd^c \log \mathfrak{B}$ is everywhere positive on M, then we call $dd^c \log \mathfrak{B} \equiv \omega_{\mathfrak{B}}$ the *Bergman metric* on M.

We would like to prove Theorem 6. We shall use the notion of bounded geometry, together with the following results, specifically Corollary 24. In fact, we only need the case Ω being a *bounded* domain in $\mathbb{C}^n$. Lemma 21 and Lemma 23 may have interests of their own. In the following, when the boundary $\partial\Omega$ of Ω is empty, i.e., $\Omega = \mathbb{C}^n$, we set $\operatorname{dist}(E, \partial\Omega) = +\infty$.

Lemma 21. *Let Ω be a domain in $\mathbb{C}^n$. Let $\{f_j\}_{j\geq 0}$ be a sequence of holomorphic functions on Ω satisfying the following property: There is a integer $N_0 \geq 0$ such that, for all $N \geq N_0$,*

$$\int_\Omega \Big| \sum_{j=0}^N c_j f_j(z) \Big|^2 dV \leq \sum_{j=0}^N |c_j|^2 \quad \text{for all } c_j \in \mathbb{C},\ 0 \leq j \leq N. \tag{5.2}$$

Then, the series

$$\sum_{j=0}^\infty f_j(z) \overline{f_j(w)} \equiv b(z,w)$$

converges uniformly and absolutely on every compact subset of $\Omega \times \Omega$. Furthermore, for every compact subset E of Ω,

$$\max_{z,w\in E} |b(z,w)| \leq \frac{C(n)}{\operatorname{dist}(E, \partial\Omega)^{2n}}, \tag{5.3}$$

where $C(n) > 0$ is a constant depending only on n.

Proof. First, suppose that $\partial\Omega$ is nonempty. We assert that, for any $z \in \Omega$,

$$\sum_{j=0}^N |f_j(z)|^2 \leq \frac{C(n)}{\operatorname{dist}(z, \partial\Omega)^{2n}} \quad \text{for all } N \geq N_0. \tag{5.4}$$

Here and below, we denote by $C(n) > 0$ a generic constant depending only on n. Assume (5.4) momentarily. By the Cauchy-Schwarz inequality,

$$\begin{aligned}\sum_{j=0}^{N} \left| f_j(z)\overline{f_j(w)} \right| &\leq \sqrt{\sum_{j=0}^{N} |f_j(z)|^2} \sqrt{\sum_{j=0}^{N} |f_j(w)|^2} \\ &\leq \frac{C(n)}{\mathrm{dist}(z, \partial\Omega)^n \cdot \mathrm{dist}(w, \partial\Omega)^n} \\ &\leq \frac{C(n)}{\mathrm{dist}(E, \partial\Omega)^{2n}},\end{aligned}$$

for all z, w in the given compact subset E and for all $N \geq N_0$. Then, letting $N \to +\infty$ yields (5.3).

To show the first statement, by the Cauchy-Schwarz inequality it is sufficient to show that the uniform convergence of $\sum_{j=0}^{\infty} |f_j(z)|^2$ on every compact subset E of Ω. (This is not an immediate consequence of (5.4), however) Let us denote by $B(z; r)$ the open ball in $\mathbb{C}^n$ centered at z of radius r. Let $\delta = \mathrm{dist}(E, \partial\Omega)/4 > 0$. Then, for each $z_0 \in E$, $B(z_0; 2\delta) \subset \Omega$. By (5.4),

$$\sum_{j=1}^{\infty} \int_{B(z_0;\delta)} |f_j(z)|^2 dV \leq \frac{C(n)}{\delta^{2n}} \mathrm{Vol}(B(z_0; \delta)) \leq C(n) < +\infty.$$

It follows that, for each $\varepsilon > 0$, there exists a constant L, depending only on ε, such that

$$\sum_{j=l}^{l+m} \int_{B(z_0;\delta)} |f_j(z)|^2 dV < \varepsilon, \quad \text{for all } l \geq L, \text{ and } m \geq 1.$$

On the other hand, applying the mean value inequality to subharmonic function $\sum_{j=l}^{l+m} |f_j(z)|^2$ on $B(z_0; \delta)$ yields

$$\sum_{j=l}^{l+m} |f_j(z)|^2 \leq \frac{C(n)}{\delta^{2n}} \int_{B(z_0;\delta)} \sum_{j=l}^{l+m} |f_j(w)|^2 dV_w.$$

Hence,

$$\sup_{B(z_0;\delta)} \sum_{j=l}^{l+m} |f_j(z)|^2 \leq \frac{C(n)}{\delta^{2n}} \varepsilon.$$

Since E can be covered by finitely many balls such as $B(z_0; \delta)$, we have proven the uniform convergence of $\sum |f_j(z)|^2$ on E.

To show (5.4), fix an arbitrary $z \in \Omega$ and $N \geq N_0$. We can assume, without loss of generality, that $|f_0(z)|^2 + |f_1(z)|^2 + \cdots + |f_N(z)|^2 \neq 0$. Denote

$$\epsilon = \frac{\mathrm{dist}(z, \partial\Omega)}{4} > 0.$$

Applying the mean value inequality to the subharmonic function $|\sum_{j=0}^N c_j f_j(z)|^2$ yields

$$\Big|\sum_{j=0}^{N} c_j f_j(z)\Big|^2 \le \frac{1}{\mathrm{Vol}(B(z;\epsilon))}\int_{B(z;\epsilon)}\Big|\sum_{j=0}^{N} c_j f_j(\zeta)\Big|^2 d\zeta$$
$$\le \frac{C(n)}{\epsilon^{2n}}\int_{\Omega}\Big|\sum_{j=0}^{N} c_j f_j(\zeta)\Big|^2 d\zeta$$
$$\le \frac{C(n)}{\mathrm{dist}(z,\partial\Omega)^{2n}}\sum_{j=0}^{N}|c_j|^2, \qquad \text{by (5.2)}.$$

Letting

$$c_j = \frac{\overline{f_j(z)}}{(|f_0(z)|^2+\cdots+|f_N(z)|^2)^{1/2}}, \quad 0\le j\le N,$$

yields (5.4). This proves the result for Ω with nonempty boundary.

If $\partial\Omega$ is empty, then $\Omega=\mathbb{C}^n$. We can replace Ω in the previous proof by a large open ball $B(0;R)$ which contains the compact subset E. The same process yields

$$\max_{z,w\in E}|b(z,w)| \le \frac{C(n)}{\mathrm{dist}(E,\partial B(0;R))^{2n}} \to 0, \quad \text{as } R\to+\infty.$$

This shows (5.3), and hence, $b\equiv 0$, for the case $\mathrm{dist}(E,\partial\Omega)=+\infty$. □

Remark 22. An example of $b(z,w)$ in Lemma 21 is the classical Bergman kernel function, for which the *equality* in (5.2) holds for *all* $N\ge 0$ and c_j, $0\le j\le N$. The arguments are well-known and standard (compare, for example, [BM48, p. 121–122]). For our applications on manifold, however, we have to state and derive the estimate under the weaker inequality hypothesis (5.2), and our estimate constant needs to be explicit on $\mathrm{dist}(E,\partial\Omega)$.

Lemma 23. *Let Ω be a domain in $\mathbb{C}^n$. Let $b(z,w)$ be a continuous function which is holomorphic in z and $\overline{w}$, and satisfies $\overline{b(z,w)}=b(w,z)$, for all $z,w\in\Omega$. If $\Omega\ne\mathbb{C}^n$, then, for each compact subset $E\subset\Omega$,*

$$\big|\partial_z^\alpha\partial_{\overline{w}}^\beta b(z,w)\big| \le \frac{C(n)\alpha!\beta!}{\mathrm{dist}(E,\partial\Omega)^{|\alpha|+|\beta|}}\max_{x,y\in E_\Omega}|b(x,y)|, \quad \text{for all } z,w\in E. \tag{5.5}$$

Here $C(n)>0$ is a constant depending only on n, α and β are multi-indices with $\alpha!\equiv\alpha_1!\cdots\alpha_n!$, $|\alpha|\equiv\alpha_1+\cdots+\alpha_n$, $\partial_z^\alpha\equiv(\partial_{z^1})^{\alpha_1}\cdots(\partial_{z^n})^{\alpha_n}$, and

$$E_\Omega=\{z\in\Omega;\mathrm{dist}(z,E)\le\mathrm{dist}(E,\partial\Omega)/2\}.$$

If $\Omega=\mathbb{C}^n$ then (5.5) continues to hold, with E_Ω replaced by any closed ball whose interior contains E.

Proof. It is sufficient to show (5.5) for the case $\Omega\subsetneq\mathbb{C}^n$; the case $\Omega=\mathbb{C}^n$ follows similarly. The inequality clearly holds when $\alpha=\beta=0$, since E is contained in E_Ω.

Consider the case $\beta = 0$ but $\alpha \neq 0$. Let $\delta = \frac{1}{4\sqrt{n}}\mathrm{dist}(E, \partial\Omega) > 0$. Pick $z, w \in E$. By Cauchy's integral formula,

$$\partial_z^\alpha b(z,w) = \frac{\alpha_1! \cdots \alpha_n!}{(2\pi\sqrt{-1})^n} \int_{\{|\zeta^1 - z^1 = \delta|\}} \cdots \int_{\{|\zeta^n - z^n| = \delta\}} \frac{b(\zeta, w)\, d\zeta^1 \cdots d\zeta^n}{(\zeta^1 - z^1)^{\alpha_1+1} \cdots (\zeta^n - z^n)^{\alpha_n+1}}$$

It follows that

$$\begin{aligned} |\partial_z^\alpha b(z,w)| &\leq \frac{C(n)\alpha!}{\delta^{|\alpha|}} \sup_{\zeta \in \mathbb{D}^n(z;\delta)} |b(\zeta, w)| \\ &\leq \frac{C(n)\alpha!}{\delta^{|\alpha|}} \max_{\zeta, y \in E_\Omega} |b(\zeta, y)|. \end{aligned} \tag{5.6}$$

Here $\mathbb{D}^n(z;\delta) \equiv \{\zeta \in \mathbb{C}^n; |\zeta^1 - z^1| < \delta, \ldots, |\zeta^n - z^n| < \delta\}$ satisfies $\mathbb{D}^n(z;\delta) \subset E_\Omega$.

Consider the general case $\alpha, \beta \neq 0$. Applying (5.6) with $b(z,w)$ replaced by $\partial_{\overline{w}}^\beta b(z,w)$ yields

$$\begin{aligned} |\partial_z^\alpha \partial_{\overline{w}}^\beta b(z,w)| &\leq \frac{C(n)\alpha!}{\delta^{|\alpha|}} \sup_{\zeta \in \mathbb{D}^n(z;\delta)} |\partial_{\overline{w}}^\beta b(\zeta, w)| \\ &= \frac{C(n)\alpha!}{\delta^{|\alpha|}} \sup_{\zeta \in \mathbb{D}^n(z;\delta)} |\partial_w^\beta b(w, \zeta)| \qquad \left(\text{since } \overline{b(\zeta,w)} = b(w,\zeta)\right) \\ &\leq \frac{\alpha!\beta! C(n)}{\delta^{|\alpha|+|\beta|}} \max_{x,y \in E_\Omega} |b(x,y)|, \qquad \text{by (5.6)}. \end{aligned}$$

Here $C(n) > 0$ denotes a generic constant depending only on n. □

Corollary 24. *Let Ω be a domain in $\mathbb{C}^n$, and let $b(z,w)$ be the function given in Lemma 21. For each compact subset $E \subset \Omega$,*

$$\left|\partial_z^\alpha \partial_{\overline{w}}^\beta b(z,w)\right| \leq \frac{\alpha!\beta! C(n)}{\mathrm{dist}(E, \partial\Omega)^{2n+|\alpha|+|\beta|}}, \quad \text{for all } z, w \in E.$$

Here $C(n) > 0$ is a constant depending only on n, and $\alpha, \beta \in (\mathbb{Z}_{\geq 0})^n$ are multi-indices, $\partial_z^\alpha \equiv (\partial_{z^1})^{\alpha_1} \cdots (\partial_{z^n})^{\alpha_n}$, $\alpha! \equiv \alpha_1! \cdots \alpha_n!$, and $|\alpha| \equiv \alpha_1 + \cdots + \alpha_n$.

Remark 25. Corollary 24 in particular implies a pointwise *interior estimate* for the Bergman kernel. This may be compared with the global estimates of Bergman kernel function in the smooth bounded domain satisfying certain boundary condition such as Bell's Condition R (see, for example, [Ker72], [BB81], and [CS01, p. 144] and references therein). Those estimates are based on the pseudo-local estimate of the $\bar{\partial}$-Neumann operator. The method here is entirely elementary, without assuming any boundary condition.

Lemma 26. *Let (M^n, ω) be a complete, simply-connected, Kähler manifold satisfy*

$$-\kappa_2 \leq K(\omega) \leq -\kappa_1 < 0 \tag{5.7}$$

for two positive constants $\kappa_2 > \kappa_1 > 0$. Let $\mathfrak{B}(z, z)$ and $\omega_{\mathfrak{B}}$ be the Bergman kernel form and Bergman metric on M. Assume that $\mathfrak{B}/\omega^n \geq c_0$ on M for some constant $c_0 > 0$. Then, $\omega_{\mathfrak{B}}$ has bounded geometry, and satisfies

$$\omega_{\mathfrak{B}} \leq C(n, c_0, \kappa_1, \kappa_2)\, \omega \quad \text{on } M, \tag{5.8}$$

where $C(n, c_0, \kappa_1, \kappa_2) > 0$ is a constant depending only on n, c_0, κ_1, and κ_2.

Proof. By (5.7) and Lemma 14 we can assume, without loss of generality, that the curvature tensor of ω and all its covariant derivatives are bounded. On the other hand, it follows from (5.7) and the standard Cartan-Hadamard theorem that, for a point $P \in M$, the exponential map $\exp_P : T_{\mathbb{R},P}M \to M$ is a diffeomorphism. This, in particular, implies that the injectivity radius of M is infinity. Thus, the manifold (M, ω) is of bounded geometry, by the second statement of Theorem 9.

Since ω has bounded geometry, there exists a constant $r > 0$, depending only on n, κ_1, κ_2, such that for each point $p \in M$, there is a biholomorphism ψ_p from the open ball $B(r) \equiv B_{\mathbb{C}^n}(0; r)$ onto its image in M such that $\psi_p(0) = p$ and $\psi_p^*(\omega)$ is uniformly equivalent to Euclidean metric on $B(r)$ up to infinite order. In particular, let $g_{i\bar{j}}$ be the metric component of $\psi_p^*(\omega)$ with respect to holomorphic coordinates $v^1, \ldots, v^n$ centered at p; then

$$C^{-1}(\delta_{ij}) \leq (g_{i\bar{j}}) \leq C(\delta_{ij}) \tag{5.9}$$

on $B(r)$. Here $B(r)$ denotes a ball in $\mathbb{C}^n$ centered at the origin of radius $r > 0$, and $C > 0$ is a generic constant depending only on κ_1, κ_2, and n.

Let $\{\phi_j\}_{j \geq 0}$ be an orthonormal basis of the Hilbert space $\mathcal{H}$ with respect to the inner product $\langle \cdot, \cdot \rangle$ given in (5.1). Then, by definition

$$\mathfrak{B}(P, Q) = \sum_{j \geq 0} \phi_j(P) \wedge \overline{\phi_j(Q)}, \qquad \text{for all } P, Q \in M.$$

Write $\phi_j = f_j(v) dv^1 \wedge \cdots \wedge dv^n$ in the chart $(B(r), \psi_p, v^j)$, for which we mean, as a standard convention, $\psi_p^* \phi_j(v) = f_j(v) dv^1 \wedge \cdots \wedge dv^n$ for $v \in B(r)$. Then, each f_j is holomorphic on $B(r)$, $j \geq 0$.

We **claim** that the domain $B(r)$ and sequence $\{f_j\}$ satisfy the requirement, (5.2), in Lemma 21. Indeed, for each $\phi \in \mathcal{H}$ with $\psi_p^* \phi = h\, dv^1 \wedge \cdots \wedge dv^n$ on $B(r)$, we have

$$\begin{aligned}
\int_{B(r)} |h(v)|^2 dV &= (-1)^{n^2/2} \int_{B(r)} |h(v)|^2 dv^1 \wedge \cdots \wedge dv^n \wedge d\bar{v}^1 \wedge \cdots \wedge d\bar{v}^n \\
&= (-1)^{n^2/2} \int_{\psi_p(B(r))} \phi \wedge \overline{\phi} \\
&\leq (-1)^{n^2/2} \int_M \phi \wedge \overline{\phi} \equiv \langle \phi, \phi \rangle.
\end{aligned}$$

Now for any $N \geq 0$ and any $c_j \in \mathbb{C}$, $0 \leq j \leq N$, substituting

$$\phi = \sum_{j=0}^{N} c_j \phi_j \quad \text{with} \quad h(v) = \sum_{j=0}^{N} c_j f_j \text{ in } B(r)$$

yields

$$\int_{B(r)} \Big| \sum_{j=0}^{N} c_j f_j(v) \Big|^2 dV \leq \langle \phi, \phi \rangle = \sum_{j=0}^{N} |c_j|^2.$$

This verifies (5.2); hence, the claim is proved. Therefore, we have

$$\psi_p^* \mathfrak{B}(v,w) = b(v,w) dv^1 \wedge \cdots \wedge dv^n \wedge d\overline{w}^1 \wedge \cdots \wedge d\overline{w}^n, \quad v, w \in B(r),$$

in which

$$b(v,w) = \sum_{j \geq 0} f_j(v) \overline{f_j(w)}$$

is a continuous function in $B(r)$, holomorphic in v and $\overline{w}$, and satisfies the interior estimate in Corollary 24. Applying Corollary 24, with $\Omega = B(r)$ and E being the closure $\overline{B(r/2)}$ of $B(r/2)$, yields

$$|\partial_v^\alpha \partial_{\overline{v}}^\beta b(v,v)| \leq \frac{\alpha! \beta! C(n)}{r^{2n+|\alpha|+|\beta|}}, \quad \text{for all } v \in \overline{B(r/2)}.$$

On the other hand, by the hypothesis $\mathfrak{B}(P,P)/\omega^n(P) \geq c_0$ for all $P \in M$; hence,

$$b(v,v) \geq c_0 \det(g_{i\bar{j}}) \geq c_0 C^{-n} > 0,$$

where (5.9) is used. Write $\omega_{\mathfrak{B}} = (\sqrt{-1}/2) g_{\mathfrak{B}, i\bar{j}} dv^i \wedge d\bar{v}^j$. Then,

$$g_{\mathfrak{B}, i\bar{j}} = b^{-1} \partial_{v^i} \partial_{\overline{v}^j} b - b^{-2} \partial_{v^i} b \, \partial_{\overline{v}^j} b$$

satisfies that

$$(g_{\mathfrak{B}, i\bar{j}}) \leq \frac{C(c_0, n)}{r^{2n+2}} (\delta_{ij}) \leq \frac{C(n, c_0, \kappa_1, \kappa_2)}{r^{2n+2}} (g_{i\bar{j}}), \tag{5.10}$$

by (5.9) again, and that

$$\Big| \partial_v^\mu \partial_{\overline{v}}^\nu g_{\mathfrak{B}, i\bar{j}} \Big| \leq \frac{C(n, c_0)}{r^{2n+2+|\mu|+|\nu|}}.$$

This proves that $\omega_{\mathfrak{B}}$ has bounded geometry. The desired inequality (5.8) (or equivalently, $\mathrm{tr}_\omega \omega_{\mathfrak{B}} \leq C$) follows from (5.10). □

Note that the hypothesis $\mathfrak{B} \geq c_0 \omega^n$ in Lemma 26 is guaranteed by the left inequality in Theorem 4 (1.1), whose local version is contained in [SY77, p. 248, line -4]. Thus, Theorem 6 follows from the left inequality of (1.1) and Lemma 26.

Remark 27. A consequence of Theorem 6 is the following technical fact on the L^2-estimate, originally proposed (conjectured) by [GW79, p. 145] to show Conjecture 5. Fix arbitrary $x \in M$ and $\eta \in T'_x M$. For any $\varphi \in \mathcal{H}$ with $\varphi(x) = 0$, define

$$\eta(\varphi) = \eta(f),$$

where φ is locally represented by $f(z) dz^1 \wedge \cdots \wedge dz^n$ near x. It is well-defined. Denote

$$\mathcal{E}_\eta(x) = \{\varphi \in \mathcal{H}; \varphi(x) = 0, \eta(\varphi) = 1\}.$$

Corollary 28. *Let (M,ω) be a simply-connected complete Kähler manifold whose sectional curvature is bounded between two negative constants $-B$ and $-A$. Then, there exists a constant $C>0$ depending only on* $\dim M$*, A, and B, such that*

$$\min_{\varphi\in\mathcal{E}_\eta(x)}\|\varphi\|\geq C,\quad \text{for any } x\in M,\ \eta\in T'_xM.$$

Corollary 28 follows immediately from Lemma 8.17 (A) and Lemma 8.19 in [GW79] and Theorem 6.

Remark 29. Theorem 6 can also be compared with a different direction, proposed by the second author, concerning the asymptotic behavior of the Bergman metric on the higher multiple mK_M of the canonical bundle for large m. The difference lies not only in the fact that M is noncompact here, but also the situation that one has to consider all terms for the case $m=1$, rather than the leading order terms for the case $m\to+\infty$.

6. Kähler-Einstein metric and holomorphic curvature

The goal of this section is to prove Theorem 3. We shall use the continuity method (Lemma 31). Theorem 3 follows immediately from Lemma 13 and Lemma 31.

The proof of Lemma 31 differs from that of Cheng-Yau [CY80] and others mainly in the complex Monge-Ampère type equation. The equation used here is inspired by the authors' work [WY16a]. This new equation is well adapted to the negative holomorphic sectional curvature and the Schwarz type lemma.

As in Cheng-Yau [CY80], we define the Hölder space $\mathcal{C}^{k,\alpha}(M)$ based on the quasi-coordinates. Let (M,ω) be a complete Kähler manifold of quasi-bounded geometry, and let $\{V_j,\psi_j\}_{j=1}^\infty$ be a family of quasi-coordinate chats in M such that

$$M=\bigcup_{j\geq 1}\psi_j(V_j).$$

Let $k\in\mathbb{Z}_{\geq 0}$ and $0<\alpha<1$. For a smooth function f on M, define

$$|f|_{\mathcal{C}^{k,\alpha}(M)}=\sup_{j\geq 1}\Big(|\psi_j^*f|_{C^{k,\alpha}(V_j)}\Big),$$

where $|\cdot|_{C^{k,\alpha}(V_j)}$ is the usual Hölder norm on $V_j\subset\mathbb{C}^n$. Then, we define $\mathcal{C}^{k,\alpha}(M)$ to be the completion of $\{f\in C^\infty(M); |f|_{\mathcal{C}^{k,\alpha}(M)}<+\infty\}$ with respect to $|\cdot|_{\mathcal{C}^{k,\alpha}(M)}$.

Lemma 30. *Let (M,ω) be an n-dimensional complete Kähler manifold of quasi-bounded geometry, and let $\mathcal{C}^{k,\alpha}(M)$ be an associated Hölder space. For any function $f\in\mathcal{C}^{k,\alpha}(M)$, there exists a unique solution $u\in\mathcal{C}^{k+2,\alpha}(M)$ satisfying*

$$\begin{cases}(\omega+dd^cu)^n=e^{u+f}\omega^n\\ C^{-1}\omega\leq\omega+dd^cu\leq C\omega\end{cases}$$

on M. Here the constant $C>1$ depends only on $\inf_M f$, $\sup_M f$, $\inf_M(\Delta_\omega f)$, n, and ω, in which $\Delta_\omega f$ denotes the Laplacian of f with respect to ω.

The proof of Lemma 30 follows from [CY80, p. 524, Theorem 4.4], with their bounded geometry replaced by the quasi-bounded geometry, which is used in the openness argument, and the bootstrap argument from the third order estimate to $\mathcal{C}^{k,\alpha}(M)$ estimate.

Lemma 31. *Let (M,ω) be an n-dimensional complete Kähler manifold such that*

$$H(\omega) \le -\kappa_1 < 0$$

for some constant $\kappa_1 > 0$. Assume that for each integer $q \ge 0$, the curvature tensor R_{m} of ω satisfying

$$\sup_{x\in M} |\nabla^q R_{\mathrm{m}}| \le B_q \tag{6.1}$$

for some constant $B_q > 0$, where ∇^q denotes the qth covariant derivative with respect to ω. Then, there exists a smooth function u on M such that $\omega_{\mathrm{KE}} \equiv dd^c \log \omega^n + dd^c u$ is the unique Kähler-Einstein metric with Ricci curvature equal to -1, and satisfies

$$C^{-1}\omega \le \omega_{\mathrm{KE}} \le C\omega, \tag{6.2}$$

where the constant $C > 0$ depends only on n and ω. Furthermore, the curvature tensor $R_{\mathrm{m,KE}}$ of ω_{KE} and its qth covariant derivative satisfies

$$\sup_{x\in M} |\nabla^q_{\mathrm{KE}} R_{\mathrm{m,KE}}| \le C_q,$$

for some constant C_q depending only on n, and $B_0, \dots, B_q$.

Proof. By hypothesis (6.1) and Theorem 9, the complete manifold (M,ω) has quasi-bounded geometry. Denote by $\mathcal{C}^{k,\alpha}(M)$ the associated Hölder space, $k \ge 0$, $0 < \alpha < 1$.

Consider the Monge-Ampère equation

$$\begin{cases} (t\omega + dd^c \log \omega^n + dd^c u)^n = e^u \omega^n, \\ c_t^{-1}\omega \le t\omega + dd^c \log \omega^n + dd^c u \le c_t\,\omega, \end{cases} \tag{MA$_t$}$$

on M with $t > 0$, where the constant $c_t > 1$ may depend on t. First, we **claim** that for a sufficiently large t, (MA)$_t$ has a smooth solution u such that

$$C^{-1}\omega \le t\omega + dd^c \log \omega^n + dd^c u \le C\omega \quad \text{on } M, \tag{6.3}$$

where $C > 0$ is a constant depending only on n and ω. To see this, note that $-dd^c \log \omega^n$ is precisely the Ricci curvature of ω. By (6.1) the curvature tensor of ω is bounded; then, for an arbitrary $t_1 > \sqrt{n} B_0$,

$$t_1\omega > -dd^c \log \omega^n \quad \text{on } M.$$

It follows that

$$t\omega + dd^c \log \omega^n > t_1\omega \quad \text{for all } t \ge 2t_1 > 0.$$

This implies that $t\omega + dd^c \log \omega^n$ defines complete Kähler metric on M; moreover, since ω is of quasi-bounded geometry, so is $t\omega + dd^c \log \omega^n$ for $t \ge 2t_1$. In particular,

$$F = \log \frac{\omega^n}{(t\omega + dd^c \log \omega^n)^n} \in \mathcal{C}^{k,\alpha}(M), \quad \text{for all } k \ge 0,\ 0 < \alpha < 1.$$

It then follows from Lemma 30 that for $t \geq 2t_1$, equation

$$(t\omega + dd^c \log \omega^n + dd^c u)^n = e^{u+F}(t\omega + dd^c \log \omega^n)^n$$

admits a solution $u \in \mathcal{C}^{k+2,\alpha}(M)$ for all $k \geq 0$ and $0 < \alpha < 1$ and satisfies (6.3). This proves the claim.

Let

$$T = \{t \in [0, 2t_1];\ \text{system } (\mathrm{MA})_t \text{ admits a solution } u \in \mathcal{C}^{k+2,\alpha}(M)\}.$$

Then T is nonempty, since $2t_1 \in T$. We would like to show T is open in $[0, 2t_1]$. Let $t_0 \in T$ with $u_{t_0} \in \mathcal{C}^{k+2,\alpha}(M)$ satisfying $(\mathrm{MA})_{t_0}$. The linearization of the operator

$$\mathcal{M}(t, v) = \log \frac{(t\omega + dd^c \log \omega^n + dd^c v)^n}{\omega^n} - v$$

with respect to v at $t = t_0$, $v = u_{t_0}$ is given by

$$\mathcal{M}_{u_{t_0}}(t_0, u_{t_0})h = \frac{d}{ds}\mathcal{M}(t_0, u_{t_0} + sh)\Big|_{s=0} = (\Delta_{t_0} - 1)h.$$

Here Δ_{t_0} denotes the Laplacian with respect to metric $\omega_{t_0} \equiv t_0\omega + dd^c \log \omega^n + dd^c u_{t_0}$. Note that $c_{t_0}^{-1}\omega \leq \omega_{t_0} \leq c_{t_0}\omega$. In particular, ω_{t_0} is complete. Furthermore, ω_{t_0} has quasi-bounded geometry up to order (k, α), that is, ω_{t_0} has quasi-coordinates satisfying (2.1), and (2.2) with the norm $|\cdot|_{C^l(U)}$ replaced by $|\cdot|_{C^{k,\alpha}(U)}$. Then, $\Delta_{t_0} - 1 : \mathcal{C}^{k+2,\alpha}(M) \to \mathcal{C}^{k,\alpha}(M)$ is a linear isomorphism, which follows from the same process as that in [CY80, pp. 520–521], with their bounded geometry replaced by the quasi-bounded geometry. Thus, T is open, by the standard implicit function theorem.

To show T is closed, we shall derive the a priori estimates. Applying the arithmetic-geometry mean inequality to the equation in $(\mathrm{MA})_t$ yields

$$ne^{u/n} \leq nt - s_\omega + \Delta_\omega u \leq C + \Delta_\omega u,$$

where $s_\omega \equiv -\mathrm{tr}_\omega dd^c \log \omega^n$ is precisely the scalar curvature of ω. Henceforth, we denote by C and C_j generic positive constants depending only on n and ω. Applying the second author's generalized maximum principle (see, for example, [CY80, Proposition 1.6]) yields

$$\sup_M u \leq C. \tag{6.4}$$

Next, observe that $(\mathrm{MA})_t$ implies

$$\mathrm{Ric}(\omega_t) = -dd^c \log \omega_t^n = -\omega_t + t\omega, \tag{6.5}$$

where $\omega_t \equiv t\omega + dd^c \log \omega^n + dd^c u > 0$. Applying [WY16a, Proposition 9] with $\omega' = \omega_t$ yields

$$\Delta' \log S \geq \Big[\frac{(n+1)\kappa_1}{2n} + \frac{t}{n}\Big]S - 1,$$

where $S = \mathrm{tr}_{\omega_t}\omega$. Again by the second author's generalized maximum principle,

$$\sup_M S \leq \frac{2n}{(n+1)\kappa_1}. \tag{6.6}$$

Combining (6.4) and (6.6) yields the estimates of u up to the complex second order (cf. [WY16a, WY16b]). In fact, by (6.6),

$$e^{-\frac{u}{n}} = \left(\frac{\omega^n}{\omega_t^n}\right)^{\frac{1}{n}} \leq \frac{S}{n} \leq \frac{2}{(n+1)\kappa_1}.$$

This implies

$$\inf_M u \geq -n \log \frac{(n+1)\kappa_1}{2}.$$

Moreover, by (6.4) we have $\sup(\omega_t^n/\omega^n) \leq C$. This together with (6.6) implies

$$\mathrm{tr}_\omega \omega_t \leq n\left(\frac{S}{n}\right)^{n-1}\left(\frac{\omega_t^n}{\omega^n}\right) \leq C.$$

Hence, $\Delta_\omega u \leq C$ and

$$\frac{(n+1)\kappa_1}{2n}\omega \leq \omega_t \leq (\mathrm{tr}_\omega \omega_t)\omega \leq C\omega. \tag{6.7}$$

One can apply [Yau78c, p. 360, 403–406] to the third order term

$$\Xi \equiv g'_{i\bar{j};k} g'_{\bar{r}s;\bar{t}} g'^{i\bar{r}} g'^{s\bar{j}} g'^{k\bar{t}}$$

to get

$$\Delta'(\Xi + C\Delta_\omega u) \geq C_1(\Xi + C\Delta_\omega u) - C_2,$$

where $g'_{i\bar{j}}$ is the metric component of ω_t and the subscript $;k$ in $g'_{i\bar{j};k}$ denotes the covariant derivative along $\partial/\partial z^k$ with respect to ω. Thus, $\sup_M \Xi \leq C$ by the second author's generalized maximum principle. Now applying the standard bootstrap argument (see [Yau78c, p. 363]) to the equation in $(\mathrm{MA})_t$ with the quasi-local coordinate charts yields $\|u\|_{\mathcal{C}^{k+2,\alpha}(M)} \leq C$. The desired closedness of T then follows immediately from the standard Ascoli-Arzelà theorem and (6.7).

Hence, we have proven $t = 0 \in T$ with $u \in \mathcal{C}^{k,\alpha}(M)$. Then, formula (6.5) tells us that $dd^c \log \omega^n + dd^c u$ is the Kähler-Einstein metric. The uniform equivalence (6.2) and boundedness of covariant derivatives of its curvature tensor follow immediately from the above uniform estimates on u. The uniqueness of complete Kähler-Einstein metric of negative curvature follows immediately from the second author's Schwarz Lemma ([Yau78a, Theorem 3]; see also [CY80, Proposition 5.5]). □

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