SOME NEGATIVELY CURVED COMPLEX GEOMETRY

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1. Introudction

In elementary complex analysis, we know that there is no bounded nonconstant holomorphic functions on the complex plane $\mathbb{C}$. By contrast, there are plenty of bounded nonconstant holomorphic functions on the unit disk $\mathbb{D}$. This phenomenon may also be interpreted in terms of geometry. That is, the complex plane admits no metric of curvature ≤ -1, while the unit disk has at least one, the Poincaré metric, given by

$$ds^2_{\mathcal{P}} = \frac{4|dz|^2}{(1 - |z|^2)^2}$$

which has curvature -1.

The geometric generalization of the above phenomenon gives one of the motivations for the current article. We roughly divide the geometric generalization into two categories: The first one is on a compact complex manifold, which is the content of Section 2. In particular, according to the current development, we shall focus on the compact Kähler manifold. The second one is on the complete noncompact Kähler manifolds; see Section 3, and we assume simply-connectedness in several cases. In the last section, Section 4, we shall discuss some open problems.

We must point out that, given the extensive study of negative complex geometry since the time of Riemann, it is impossible for this short article to include all the developments. Based on our biased preferences, we narrow down to discuss the recent development of several questions and conjectures raised in the 1970s, which are the driving force of our own work. However, even for these directions, our bibliography is by no means complete.

2. Negative holomorphic curvature on compact Kähler manifolds

Let us generalize the fact that the unit disk admits plenty of holomorphic functions to the situation of a compact complex manifold. Since the manifold is compact, there

The first author was partially supported by the NSF grant DMS-1611745. The second author was partially supported by the NSF grant DMS-1308244 and DMS-1607871, and the Simonyi Endowment of the Institute for Advanced Study.

is no bounded nonconstant holomorphic function. However, a natural candidate to replace a holomorphic function is the holomorphic section of the canonical bundle K_M over M. Such a holomorphic section is locally given by

$$f(z)dz^1 \wedge \cdots \wedge dz^n,$$

where $n = \dim_{\mathbb{C}} M$, and the coefficient f is a holomorphic function defined on a local coordinate chart. In particular, when M is a domain of $\mathbb{C}^n$, the holomorphic section of K_M is a one-to-one correspondence to the holomorphic function on M.

More generally, one can consider the holomorphic section of the mth product mK_M of the canonical bundle K_M, where m is a positive integer. Its holomorphic section locally is just $f(z)(dz^1 \wedge \cdots \wedge dz^n)^m$ in which f is a locally defined holomorphic function. Again if M is a domain of $\mathbb{C}^n$, the holomorphic section of mK_M one-to-one corresponds to the holomorphic function on M. A nature question is as below.

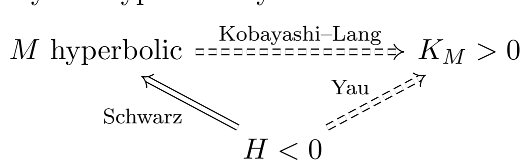
Question 2.1. *When does mK_M have plenty of holomorphic sections?*

A classical result of Kodaira tells us that, if K_M is positive (in the sense that there is a metric on K_M such that the curvature form is positive definite), then K_M is ample; in particular, for all sufficiently large m, the bundle mK_M have lots of holomorphic sections. Now the question becomes

Question 2.2. *When does a complex manifold M have $K_M > 0$?*

A well-known theorem of S. S. Chern says that if M admits a metric with negative Ricci curvature, then $K_M > 0$. In view of the resolution of Calabi conjecture in the case of negative scalar curvature, independently by the second author and Aubin, we know that $K_M > 0$ if and only if M admits a Kähler-Einstein metric with negative scalar curvature. On the other hand, in the 1970s there were considerable research activities concerning the holomorphic maps. This direction naturally leads to the above questions.

The second author has conjectured that if a compact complex manifold M has negative holomorphic sectional curvature, then $K_M > 0$. Moreover, a conjecture of Lang asserts that if M is projective and Kobayashi hyperbolic, then $K_M > 0$, while the Kobayashi conjecture extends Lang's conjecture to a compact Kähler manifold. All three conjectures would continue to hold for submanifolds provided they hold for the ambient manifolds, because of the decreasing property of holomorphic sectional curvature and the Kobayashi hyperbolicity.



These conjectures are related via the Schwarz Lemma: If M has negative holomorphic curvature, then M is Kobayashi hyperbolic. Its converse is not true in general, in view of the hyperbolic surface constructed by Demailly [Dem97, Theorem 8.2].

Several authors have made contributions to these conjectures. In complex dimension two, these are answered affirmatively independently by Bun Wong [Won81] and Campana [Cam91], by means of the classification theory of compact complex surfaces. A short direct proof is later provided by the join paper of Godon Heier, Steven Lu, and Bun Wong [HLW10] using only standard algebraic geometry (the Nakai-Moishezon-Kleiman criterion, the Riemann-Roch theorem, and the Hodge index theorem) and a generalized Gauss-Bonnet theorem due to Bishop-Goldberg.

In higher dimensions, it is natural to first consider the projective algebraic manifolds, where some algebraic-geometric tools and partial classifications are available. Peternell [Pet91] proves the Kobayash-Lang conjecture for projective three-fold except for the Calabi-Yau threefold which contain no rational curves.

As a testing case, together with Pit-Mann Wong, we prove several years ago the second author's conjecture for all projective manifolds with Picard number equal to one [WWY12]. The holomorphic sectional curvature in [WWY12] is only assumed to be *quasi-negative*, i.e., nonpositive everywhere and negative at one point. The quasi-negativity of holomorphic curvature may not be stronger than the Kobayashi hyperbolicity, as the projective manifold of quasi-negative holomorphic curvature may contain elliptic curves.

The projective threefold case of the second author's conjecture has been completely settled by a series of papers of Heier-Lu-Wong [HLW10, HLW16], which make an interesting connection to the abundance conjecture in the algebraic geometry. They indeed prove the second author's conjecture by assuming the validity of the abundance conjecture, which is known to hold for dimension less than four.

More precisely, Heier-Lu-Wong prove that if a projective manifold with negative holomorphic sectional curvature then the canonical bundle is nef, and the nef dimension is equal to the dimension of the manifold. A version of the abundance conjecture asserts that for a projective manifold with nef canonical bundle, the Kodaira dimension is equal to the nef dimension, that is, the manifold is of general type. Since the manifold contains no rational curve, the canonical bundle is ample.

We remove the need for the abundance conjecture in [WY16a]. In fact, [WY16a] provides two slightly different proofs for the second author's conjecture for the projective manifolds in all dimensions. That is, if a projective manifold M admits a Kähler metric with negative holomorphic sectional curvature, then K_M is ample. Furthermore, every smooth subvariety in M also has ample canonical bundle, in view of the decreasing property of holomorphic sectional curvature. In particular, every nonsingular subvariety of a smooth compact quotient of the unit ball in $\mathbb{C}^n$ has ample canonical bundle.

The first proof in [WY16a] reduces to show the integral inequality

$$\int_M c_1(K_M)^n > 0.$$

In fact, the hyperbolicity implies M contains no rational curve; by Mori's theory, K_M is nef. The nefness together with the integral inequality implies K_M is big, which is due to the result of Demailly, Siu, Trapani, and other people, as an application of Demailly's holomorphic Morse inequality.

An important step in [WY16a] is to introduce a Monge-Ampère type equation (see (2.1)) to construct a family of Kähler metrics whose Ricci curvature has a uniform lower bound. This allows one to apply the refined Schwarz Lemma to show the desired integral inequality. The refined Schwarz Lemma is initiated by Ahlfors [Ahl38], developed by Chern [Che68], the second author [Yau78a], Royden [Roy80], and many other people. The version used in [WY16a] is a strengthen result of our previous work joint with Fangyang Zheng [WYZ09], and with Pit-Mann Wong [WWY12].

The Monge-Ampère type equation (2.1) may be compared with the equation of the form $(\omega + t\Phi + dd^c u)^n = \gamma(t)\omega^n$ constructed earlier in [WYZ09], where Φ is a given $(1,1)$ form, γ is given a smooth function on $t \in \mathbb{R}$, and we want to solve u for $t = 1$. The latter equation was used to solve a smooth representative for a given nef cohomology class on a compact Kähler manifold of nonnegative quadratic bisectional curvature. The equation in [WYZ09] is later bypassed by X. Zhang [Zha12], Chau-Tam [CT12], and others.

The second proof in [WY16a] is to directly show that the family of metrics converges to a Kähler-Einstein metric, which implies $K_M > 0$. The second proof again make uses of the Monge-Ampère type equation (2.1), and the refined Schwarz lemma. The difference is that we can derive all uniform estimates with constants depending only on the background metric.

By using both the Monge-Ampère equation and the refined Schwarz Lemma in [WY16a], Tosatti-Yang [TY17] show that if a Kähler manifold has nonpositive holomorphic sectional curvature, the canonical bundle is nef. This combining the second proof in Wu-Yau [WY16a] enable them to extend our result to the Kähler manifolds. In [WY16b] we provide a direct proof of the second author's conjecture in the Kähler case, by modifying the second proof in [WY16a]. This proof uses purely geometric analysis, bypassing the notion of nefness. The proof will be given below.

It is natural to extend these results to the case the holomorphic sectional curvature H is quasi-negative, as in Wong-Wu-Yau [WWY12]. This extension is established by Diverio-Trapani [DT19] and Wu-Yau [WY16b], again using the Monge-Ampère type equation and the refined Schwarz lemma. In this situation, the key is the compactness argument. Diverio-Trapani uses the pluripotential theory, while we use an elementary lemma inspired by the work of S. Y. Cheng and the second author [CY75]. There are other extensions or related work (see for example [HLWZ18], [YZ19], [Cad], [FX], [Nom18], [Gue18], [Lee18]).

We now summarize some of the recent results and provide a unifying proof below.

Theorem 2.3 ([WY16a], [TY17], [DT19], [WY16b])**.** *Let (M, ω) be a compact Kähler manifold, and $H(\omega)$ be the holomorphic sectional curvature of ω.*

(i) *If $H(\omega) < 0$ everywhere on M, then $K_M > 0$.*
(ii) *If $H(\omega) \leq 0$ everywhere on M, then K_M is nef.*
(iii) *If $H(\omega)$ is quasi-negative, i.e., $H(\omega) \leq 0$ everywhere and $H(\omega) < 0$ at one point of M, then $K_M > 0$.*

First to show (i). Inspired by the nefness of K_M, we consider

$$\begin{cases} (t\omega + dd^c \log \omega^n + dd^c u)^n = e^u \omega^n, \quad t \geq 0, \\ \omega_t \equiv t\omega + dd^c \log \omega^n + dd^c u > 0. \end{cases} \tag{2.1}$$

Here ω is the background Kähler metric with negative holomorphic sectional curvature, and $dd^c \log \omega^n = -\mathrm{Ric}(\omega)$ is the Chern form representing the first Chern class of K_M.

We would like to solve u for $t = 0$, by the continuity method. First, we claim that for a sufficiently large t_1, the equation has a smooth solution. (This is indeed an important step which enables us to bypass the nefness.) To see this, we can pick a large t_1 such that $t_1\omega + dd^c \log \omega^n$ is positive definite on the compact manifold. Then $t\omega + dd^c \log \omega^n$ defines a Kähler metric since it is d-closed. Note that the equation can be rewritten as

$$(t_1\omega + dd^c \log \omega^n + dd^c u)^n = e^{u+f}(t_1\omega + dd^c \log \omega^n)^n,$$

where f is a smooth function given by

$$f = \log \frac{\omega^n}{(t_1\omega + dd^c \log \omega^n)^n}.$$

This equation has a smooth solution u, by the early work of the second author on the Calabi conjecture [Yau78c]. This proves the claim.

Let

$$I = \{t \in [0, t_1]; \omega_t^n = e^u \omega^n, \omega_t > 0\}.$$

Then I is not empty, since $t_1 \in I$. To see I is open in $[0, t_1]$, let $t_0 \in I$ with solution u_{t_0}. Define

$$\mathcal{M}(t, v) = \log \frac{(t\omega + dd^c \log \omega^n + dd^c v)^n}{\omega^n} - v$$

for all (t, v) in a near (t_0, u_{t_0}). Then $\mathcal{M}(t_0, u_{t_0}) = 0$. Note that the linearization of $\mathcal{M}$ at (t_0, u_{t_0}) with respect to v is precisely given by

$$\Delta_{\omega_{t_0}} - 1,$$

which is invertible between the Hölder spaces. Thus, applying the implicit function theorem yields the openness of I in $[0, t_1]$.

The closedness of I requires the Schwarz Lemma. We use the following version of Schwarz Lemma, which is, in turn, based on [Yau78a, Roy80, WYZ09, WWY12].

Lemma 2.4 ([WY16a])**.** *Let M^n be a complex manifold with two Kähler metrics ω_1 and ω_2. If $H(\omega_1) \le -\kappa$ and $\mathrm{Ric}(\omega_2) \ge \lambda\omega_2 + \mu\omega_1$, then*

$$\Delta_{\omega_2} \log(\mathrm{tr}_{\omega_2}\omega_1) \ge \Big(\frac{(n+1)\kappa}{2n} + \frac{\mu}{n}\Big)\mathrm{tr}_{\omega_2}\omega_1 + \lambda.$$

Here λ, κ, μ are continuous functions on M and $\kappa \ge 0$, $\mu \ge 0$ on M.

The key feature of the Monge-Ampère equation is that $\omega_t^n = e^u\omega^n$ implies

$$\mathrm{Ric}(\omega_t) = -\omega_t + t\omega, \quad \sup u \le C.$$

Since $H(\omega) < 0$ and M is compact, $H(\omega) \le -\kappa$ for some constant $\kappa > 0$.

Applying the Schwarz lemma $\omega_1 = \omega$ and $\omega_2 = \omega_t$ yields

$$\Delta_{\omega_2} \log(\mathrm{tr}_{\omega_t}\omega) \ge \Big(\frac{(n+1)\kappa}{2n} + \frac{t}{n}\Big)\mathrm{tr}_{\omega_t}\omega - 1.$$

By the maximum principle,

$$\mathrm{tr}_{\omega_t}\omega \le \frac{2n}{(n+1)\kappa}.$$

The estimates on $\mathrm{tr}_{\omega_t}\omega$ and $\sup u$ are sufficient for the closedness of I, by the argument in [Yau78c]. One way to see this is as below. We can normalize at one point such that the components of ω satisfies $g_{i\bar{j}} = \delta_{ij}$ and the components of ω_t satisfies $g'_{i\bar{j}} = \lambda_i\delta_{ij}$. The bound on $\sup u$ yields

$$\lambda_1 \cdots \lambda_n \le C$$

while the bound on $\mathrm{tr}_{\omega_t}\omega$ is

$$\mathrm{tr}_{\omega_t}\omega = \frac{1}{\lambda_1} + \cdots + \frac{1}{\lambda_n} \le C.$$

By the elementary inequality

$$\sum_{i=1}^n \lambda_i \le (\mathrm{tr}_{\omega_t}\omega)^{n-1} \prod_{i=1}^n \lambda_i \le C.$$

Hence,

$$C^{-1} \le \lambda_i \le C, \quad i = 1, \ldots, n.$$

This also gives a lower bound on u, as

$$e^u = \lambda_1 \cdots \lambda_n \ge C^{-n}.$$

Thus, we obtain

$$C^{-1}\omega \le \omega_t \le C\omega,$$

as well as the estimates of u up to the second order.

The third order estimate of u can be derived in a similar way as in my early work: Let $Y \equiv g'_{i\bar{j};k} g'_{\bar{r}a;\bar{b}} g'^{i\bar{r}} g'^{a\bar{j}} g'^{k\bar{b}}$ to get

$$\Delta_{\omega_t}(Y + C\Delta_\omega u) \ge C_1(Y + C\Delta_\omega u) - C_2.$$

Hence, by the maximum principle,

$$Y \leq C.$$

Now letting $t \to 0$ we get a smooth solution u_* satisfying

$$\begin{aligned}(dd^c \log \omega^n + dd^c u_*)^n &= e^{u_*}\omega^n, \\ dd^c \log \omega^n + dd^c u_* &> 0,\end{aligned}$$

which gives the desired Kähler-Einstein metric with negative scalar curvature. This in particular implies the canonical bundle is positive. Thus, statement (i) is proved.

Next to show (ii). It is sufficient to solve the Monge-Ampère type equation

$$\omega_t^n = (t\omega + dd^c \log \omega^n + dd^c u)^n = e^u \omega^n, \quad \omega_t > 0,$$

for every small $t > 0$. Again use the continuity method: The nonemptyness, openness, and C^0 estimate are the same as (i). Only difference is that the upper bound κ of $H(\omega)$ can be zero. Now the Schwarz Lemma reads

$$\Delta_{\omega_t} \log(\mathrm{tr}_{\omega_t}\omega) \geq \frac{t}{n}\mathrm{tr}_{\omega_t}\omega - 1.$$

It follows that

$$\mathrm{tr}_{\omega_t}\omega \leq \frac{n}{t} \leq \frac{n}{t_2} \quad \text{for all } t_2 \leq t \leq t_1.$$

Here $t_2 > 0$ is arbitrary. The C^2 estimate becomes

$$t_2 C^{-1}\omega \leq \omega_t \leq C t_2^{1-n}\omega.$$

The higher order estimates of u depends on t_2. Since $t_2 > 0$ is arbitrary, we obtain a smooth solution u of $\omega_t^n = e^u\omega^n$. In particular, $e^u\omega^n$ gives rise to a smooth metric on K_M so that its curvature form

$$dd^c \log(e^u\omega^n) = \omega_t - t\omega > -t\omega.$$

This implies K_M is nef. This proves (ii).

Let us now prove (iii). Since $H(\omega) \leq 0$ implies that K_M is nef and M contains no rational curve, it is sufficient to show K_M is big (this is indeed based on Demailly's fundamental work on the Morse inequality), i.e.,

$$\int_X c_1(K_M)^n > 0.$$

Note that

$$\int_X \omega_t^n = \int_X c_1(K_M)^n + tn\int_X c_1(K_M)^{n-1} \wedge \omega + O(t^2), \quad t \to 0.$$

It suffices to find a sequence t_j such that

$$\lim_{j\to+\infty} \int_X \omega_{t_j}^n > 0.$$

We have to bound $\max u_{t_j}$ away from $-\infty$.

The following lemma is inspired by the joint work of the second author with S. Y. Cheng [CY75].

Lemma 2.5 ([WY16b])**.** *Let (M,ω) be an n-dimensional compact Kähler manifold, and let v be a C^2 function satisfying $v \le -1$ on M and*

$$\Delta_\omega v \ge -C_0$$

for some constant $C_0 > 0$ on M. Then,

$$\int_M |\log(-v)|^2 \omega^n + \int_M |\nabla \log(-v)|^2 \omega^n \le C\Big[1 + \min_M(-v)\Big]$$

where $C > 0$ is a constant depending only on n, ω, and C_0.

Assume Lemma 2.5 momentarily. Let us proceed to complete the proof of (iii). Note that $\mathrm{tr}_\omega \omega_t > 0$ implies

$$\Delta_\omega u \ge -nt + s \ge -C_0.$$

Thus, the hypothesis in Lemma 2.5 is satisfied. Then, Lemma 2.5 allows us to apply the Rellich compactness lemma. Apply the compactness lemma to $v_t = u_t - \max u_t - 1$ to obtain a sequence $\log(-v_{t_j})$ converges in $L^q(M)$ to w. Thus,

$$v_{t_j} \longrightarrow -e^w \quad \text{almost everywhere on } M.$$

Applying Schwarz lemma and elementary inequality to obtain

$$\Delta_{\omega_{t_j}} \log(\mathrm{tr}_{\omega_{t_j}} \omega) \ge \frac{(n+1)\kappa}{2} e^{-\frac{\max u_{t_j}}{n}} - 1.$$

Integrating against $\omega_{t_j}^n$ yields

$$\exp\Big(-\frac{\max u_{t_j}}{n}\Big) \le \frac{2\int_M e^{v_{t_j}} \omega^n}{(n+1)\int_M \kappa e^{v_{t_j}} \omega^n} \le C,$$

since $\kappa > 0$ in an open subset of M. This gives the desired uniform bound on $\max u_{t_j}$.

By passing to a subsequence we can assume $u_{t_l} \to -e^w + c$ almost everywhere in M. Hence,

$$\lim_{l\to+\infty} \int_M \omega_{t_l}^n = \lim_{l\to+\infty} \int_M e^{u_{t_l}} \omega^n > 0.$$

This completes the proof of the result that $H(\omega)$ being quasi-negative implies K_M is ample.

One way to see Lemma 2.5 is as below. We compute

$$\Delta_\omega \log(-v) = \frac{-\Delta_\omega v}{-v} - |\nabla \log(-v)|^2.$$

Since $\Delta_\omega v \ge -C_0$ and $\min_M(-v) \ge 1$, integrating over M yields

$$\int_M |\nabla \log(-v)|^2 \le C_0 \int_M \omega^n.$$

On the other hand, applying the weak Harnack inequality (see [GT01, p. 194, Theorem 8.18] for example) to $\Delta_\omega(-v) \le C_0$ yields that, for any $1 \le q < n/(n-1)$,

$$\Big(\int_M (-v)^q \omega^n\Big)^{1/q} \le C\Big[1 + \min_M(-v)\Big]. \tag{2.2}$$

In particular, put $q = 1$ and note $(-v) = e^{\log(-v)} \ge [\log(-v)]^2/2$. This implies the L^2 norm of $\log(-v)$. Combining these two inequalities yields the desired estimate.

Here is a technical remark. Note that the estimate for $q = 1$ in (2.2) is sufficient for our purpose here. For this case, one can also make use of positive Green's function as in [Yau78c, p. 352] (compare [GW16]). Let $v(x_0) = \max_X v$. Apply the positive Green's function with respect to ω yields

$$\begin{aligned}\min_M(-v) = -v(x_0) &= \frac{1}{\mathrm{Vol}(M)}\int_M(-v) + \int_M G(x_0,x)\Delta_\omega v(x)\\ &\ge \frac{1}{\mathrm{Vol}(M)}\int_M(-v) - C.\end{aligned}$$

This implies (2.2) for $q = 1$.

3. Invariant metrics on negatively pinched complete Kähler manifolds

We seek generalize the positivity to complete noncompact Kähler manifolds. This is in fact the direct higher dimensional analogue of the phenomenon of the unit disk mentioned in the introduction. In the general situation we replace the unit disk by a *simply connected* complete Kähler manifold whose sectional curvature is bounded above by a negative constant. Up to this moment, it is still a *conjecture* that such a manifold admits one bounded nonconstant holomorphic function (cf. [Yau82a, p. 678, Problem 38]). (Of course, if it is true then the manifold would possess many bounded nonconstant holomorphic functions.) We remark that the simply-connectedness assumption is needed, in view of the example of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$.

We can still generalize the positivity to the complete noncompact Kähler manifolds in the following sense. Recall that for a compact Kähler manifold M, the ampleness of the canonical bundle K_M is equivalent to the existence of Kähler-Einstein metric on M with negative scalar curvature. Thus, we can characterize the positivity of K_M on a complete noncompact manifold by the existence of a complete Kähler-Einstein metric of negative scalar curvature. Our first result in this direction is given below.

Theorem 3.1 ([WY17])**.** *Let (M,ω) be a complete Kähler manifold whose holomorphic sectional curvature $H(\omega)$ satisfies $-\kappa_2 \le H(\omega) \le -\kappa_1$ for two constants $\kappa_1, \kappa_2 > 0$. Then, M admits a unique complete Kähler-Einstein metric ω_{KE} with Ricci curvature equal to -1. Furthermore, ω_{KE} is uniformly equivalent to ω, and the curvature tensor of ω_{KE} and all its covariant derivatives are bounded.*

Theorem 3.1 differs from the previous result on complete Kähler-Einstein metrics such as [CY80] and [TY87] in that it essentially assumes no condition on $\mathrm{Ric}(\omega)$. Theorem 3.1 generates new examples of complete Kähler-Einstein manifolds (see [WY18, Section 5]). For instance, every closed submanifold Σ in a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ admits a complete Kähler metric with negatively pinched holomorphic sectional curvature, by virtue of the distance decreasing property of H and [Kle78, p. 279, Corollary 1]. Then, by Theorem 3.1, the manifold Σ admits a complete Kähler-Einstein metric with Ricci curvature equal to -1.

The proof again makes use of the Monge-Ampère equation in the proof of Theorem 2.3. A major difference is that, on the complete noncompact manifold we need an effective version of the quasi-bounded geometry, by which we can adapt the Schauder type estimate to handle the nonemptyness, openness, and the bootstrap argument in the continuity method. We shall describe the idea shortly.

Notice that the complete Kähler-Einstein metric of negative scalar curvature is one of the classical invariant metrics on a complex manifold. An *invariant metric* is a metric (or a length function) L on a complex manifold M such that every biholomorphic map F gives an isometry $F^*L = L$. Thus, an invariant metric depends only on the complex structure of the complex manifold. Besides the Kähler-Einstein metric of negative scalar curvature, the classical invariant metrics also include the Bergman metric, the Carathéodory-Reiffen metric, and the Kobayashi-Royden metric (see [Wu93] for example).

The invariant metrics are closely related to the complete Kähler manifold with negative curvature. Let us recall some conjectures. A conjecture which is stronger than the one given above states the following: *If a simply-connected complete Kähler manifold has sectional curvature bounded above by a negative constant, then the manifold is biholomorphic to a bounded domain in* $\mathbb{C}^n$ (see [SY77, p. 225] and [Wu83, p. 98]; compare [Wu67, p. 195, (1)] and [Yau82b, p. 47, c.]. This problem has in fact been proposed by the second author back in 1971). On the other hand, it has been well-known that on a bounded strictly pseduoconvex domain in $\mathbb{C}^n$ with smooth boundary, the four classical invariant metrics exist and are all quasi-isometric to each other; see for example [Die70, Gra75, CY80, Lem81, BFG83, Wu93]. The quasi-isometries do not extend to bounded weakly pseudoconvex domains with smooth boundary, however; see [DFH84] for example.

On Kähler manifolds, R. E. Greene and H. Wu have posted two notable conjectures. Their first conjecture concerns the Kobayashi-Royden metric. Let us recall the definition. Let M be a complex manifold and $T'M$ the holomorphic tangent bundle. For each $x \in M$ and $\eta \in T'_x M$, consider a holomorphic map ϕ from the unit disk to M such that $\phi(0) = x$ and $\phi_*(v) = \eta$. The Kobayshi-Royden metric $\mathfrak{K}(x, \eta)$ is define to be the infimum of the Euclidean norm $|v|_0$ over all such maps ϕ.

The Greene-Wu conjecture states that, *if a simply-connected complete Kähler manifold has sectional curvature bounded between two negative constants, then the Kobayashi-Royden metric is quasi-isometric to the background Kähler metric* [GW79, p. 112]. In fact, it is well-known that, due to the Schwarz Lemma, the Kobayashi-Royden metric is always bounded below by a hermitian metric, provided the holomorphic sectional curvature of the hermitian metric is bounded above by a negative constant. Thus, it is the upper bound of Kobayashi-Royden metric that requires a proof.

By using the quasi-bounded geometry develop in the proof of Theorem 3.1, we are able to prove a stronger result, which removes the simple connectedness, and relaxes the sectional curvature by the holomorphic sectional curvature in the Greene-Wu conjecture. In particular, the manifold in Theorem 3.2 does not have to be Stein.

Theorem 3.2 ([WY17])**.** *If a complete Kähler manifold with holomorphic sectional curvature bounded between two negative constants, then its Kobayshi-Royden metric is uniformly equivalent to the background Kähler metric.*

We remark that Greene-Wu [GW79, p. 112] proposed to show their conjecture by studying the extremal holomorphic map $\phi : \mathbb{D} \to M$ which realizes the infimum in the definition of the Kobayashi-Royden metric $\mathfrak{K}$. Up to this date, it is still lack of a good knowledge on the extremal map.

Our approach to Theorem 3.2 can be motivated by the interior estimate given below. For any $x \in M$ and $\xi \in T'_x M$, applying the mapping decreasing property to the coordinate chart $(B(r) \subset \mathbb{C}^n, \psi)$ with $\psi(0) = x$ and $\psi_*(v) = \xi$ yields

$$\mathfrak{K}_M(x, \xi) \leq \mathfrak{K}_{\psi(B(r))}(x, \xi) \leq \mathfrak{K}_{B(r)}(0, v) = \frac{|v|_{\mathbb{C}^n}}{r}.$$

Then, what we expect to have are the properties that

$$|v|_{\mathbb{C}^n} \leq C \psi^* \omega(0, v) = C|\xi|_\omega,$$

or more generally,

$$C^{-1} \omega_{\mathbb{C}^n} \leq \psi^* \omega \leq C \omega_{\mathbb{C}^n},$$

and that the constants $C, r > 0$ depend only on the curvature bounds. These properties lead us naturally to the quasi-bounded geometry, for which we shall discuss below.

Greene-Wu also conjectures on the Bergman metric. Let us recall the definition. Let M be an n-dimensional complex manifold. There is a natural inner product on the smooth $(n, 0)$ forms given by

$$\langle \varphi, \psi \rangle = (-1)^{n^2/2} \int_M \varphi \wedge \overline{\psi}.$$

This inner product is independent of the hermitian metrics on M and on K_M.

Denote by $\mathcal{H}$ the set consisting of holomorphic n-forms φ such that the induced norm $\|\varphi\| < +\infty$. Then $\mathcal{H}$ is a separable Hilbert space. Assume $\mathcal{H} \neq \{0\}$. Then $\mathcal{H}$

contains an orthonormal basis $\{e_j\}_{j\geq 0}$ with respect to the inner product. One can define an (n,n) form $\mathfrak{B}$ on $M \times M$ by

$$\mathfrak{B}(p,q) = \sum_{j\geq 0} e_j(p) \wedge \overline{e_j}(q).$$

This definition is independent of the choice of orthonormal basis. Along the diagonal of $M \times M$, we can express $\mathfrak{B}(p,p)$ in terms of local coordinates $(z^1, \dots, z^n)$ as

$$\mathfrak{B}(z,z) = b(z,z) dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n.$$

We call $\mathfrak{B}(p,p)$ and $b(z,z)$ the *Bergman kernel form* and *Bergman kernel function* on M, respectively. When M is a domain in $\mathbb{C}^n$, the Bergman kernel function b recovers the classical Bergman kernel. We further assume that the Bergman kernel form $\mathfrak{B} > 0$ everywhere on M. Let

$$\omega_{\mathfrak{B}} = dd^c \log b,$$

which is globally defined on M. We call $\omega_{\mathfrak{B}}$ the *Bergman metric* on M, if $dd^c \log b > 0$ everywhere on M. By contrast to those Bergman metrics defined via a general positive line bundle, the Bergman metric given here is an invariant metric on M.

Based on [SY77], Greene-Wu [GW79, p. 144] proves the following result: If (M, ω) is complete, simply-connected, Kähler manifold whose sectional curvature is pinched between two negative constants, then M admits a Bergman metric $\omega_{\mathfrak{B}}$, which dominates ω, i.e., $\omega_{\mathfrak{B}} \geq C\omega$ on M. In particular, $\omega_{\mathfrak{B}}$ is complete. Then, Greene-Wu conjectures that the Bergman metric $\omega_{\mathfrak{B}}$ is also dominated by ω; in other words, $\omega_{\mathfrak{B}}$ is quasi-isometric to ω [GW79, p. 145].

We remark that, as far as the existence of a complete Bergman metric is concerned, the curvature condition can be relaxed to the sectional curvature $\leq A/r^2$; see for example [CZ02] and references therein. On the other hand, for this conjecture, the simply-connectedness is needed, as we have examples of $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ and the punctured disk.

This conjecture of Greene-Wu would follow immediately if one can derive an volume estimate $\omega_{\mathfrak{B}}^n \leq C\omega^n$. However, the volume estimate is not trivial: The Schwarz lemma does not apply, as the curvatures of Bergman metric do not have a sign in general. As another attempt, Greene-Wu [GW79, p. 145] proposed to show the following technical statement: For every $x \in M$, there is a uniform positive lower bound for $\|\varphi\|$ where φ runs over all square integrable holomorphic n-forms such that φ vanish at x of order 1. However, such an estimate seems no easier than the conjecture itself, and it is now a consequence of our next result.

We take a different approach, using the bounded geometry to derive the pointwise interior estimate. This enables us to prove the conjecture of R. E. Greene and H. Wu.

Theorem 3.3 ([WY17])**.** *If (M, ω) is a complete, simply-connected, Kähler manifold whose sectional curvature is bounded between two negative constants $-A$ and $-B$,* *then its Bergman metric $\omega_{\mathfrak{B}}$ has bounded geometry and satisfies*

$$\omega_{\mathfrak{B}} \leq C\omega \quad \text{on } M,$$

where the constant $C > 0$ depends only on A, B, and $\dim M$. Consequently, $\omega_{\mathfrak{B}}$ is quasi-isometric to ω.

The following result follows immediately from Theorem 3.1, Theorem 3.2, and Theorem 3.3.

Corollary 3.4. *On a complete, simply-connected, Kähler manifold M, the three classical invariant metrics, Kähler-Einstein, Bergman, and Kobayashi-Roydent metrics exist and are all quasi-isometric to each other.*

We in fact provide certain unifying approach to these three seemingly totally different metrics. Let us consider for example the Kähler-Einstein metric of negative scalar curvature. As in the compact case, we employ the Monge-Ampère equation

$$\begin{cases} (t\omega + dd^c \log \omega^n + dd^c u)^n = e^u \omega^n, \\ \omega_t \equiv t\omega + dd^c \log \omega^n + dd^c u > 0 \end{cases} \tag{MA$_t$}$$

with the continuity method. The key difference is below: Notice that the openness, nonemptyness, and bootstrap argument use the Schauder type estimate. The standard Schauder estimate requires the injectivity radius of the complete manifold to be positive, for which (M, ω) need not have. An example is the Poincaré punctured disk.

To overcome this difficulty, we need to develop the notion of (quasi-) bounded geometry initiated by the second author with S. Y. Cheng [CY80]. Let us begin with the Riemannian version of quasi-bounded geometry, pointed out by the second author in 1980 ([WL97, Appendix]), as an extension of the bounded geometry [CY80, Section 8]. The idea goes as follows: If the curvature tensor of the Riemannian manifold (M, ω) is bounded, then there is a constant $R > 0$, depending only on the sectional curvature upper bound, such that for any point $x \in M$, the exponential map $\exp_x$ is immersion on the ball $B(R)$ of radius R in the tangent space. Then, the pullback metric on $B(R)$ under $\exp_x$ has a nice property that its Laplacian is uniformly elliptic on the ball $B(R)$. If, in addition, the curvature tensor of ω and all its derivatives are bounded on M, we can apply the Schauder estimates to the Laplacian of $\exp_x^* \omega$ on $B(R)$.

Thus, instead of the usual coordinate charts, we shall work on the quasi-coordinate charts $\{(B(R), \exp_x)\}$, which is sufficient for solving partial differential equations on manifolds. However, for our further applications on invariant metrics, it is desired to have holomorphic coordinate charts $\{(B(R_1), \psi_x)\}$ for which the radius R_1 is uniformly bounded away from zero.

Notice that the exponential map is in general not holomorphic, if the tangent space is endowed with the standard complex structure of $\mathbb{C}^n$. Nevertheless, we can pullback the complex structure from the manifold to the tangent space via the exponential map. Then, the exponential map is holomorphic, but the geodesic normal coordinates need not be holomorphic. To produce holomorphic coordinates, one needs to solve a $\bar{\partial}$-equation.

The starting point is the following inequality, established by the second author with Y. T. Siu [SY77],

$$|\bar{\partial}(x^j + \sqrt{-1}\, x^{n+j})| \le Cr^2 \quad \text{on } B(R),$$

where $x = (x^1, \ldots, x^{2n})$ is a geodesic normal coordinate system and $r = |x|$ is the Euclidean distance. The Siu-Yau inequality allows one to use the singular weight in the L^2 estimate of $\bar{\partial}$. Then, we obtain a system of holomorphic functions which form an independent set at the origin, thanks to the singular weight. By the implicit function theorem, these holomorphic functions form a coordinate system in a small ball $B(\delta)$ of the origin (cf. [SY77, pp. 247–248], [GW79, pp. 160–161], and [TY90, p. 582]).

A subtlety is that the radius δ may a priori depend on the point x. To see this, note that the complex structure on $B(R)$ is not the standard one inherited from $\mathbb{C}^n$ but the one pullback from the complex manifold via $\exp_x$. Thus, for different x, the corresponding $B(R)$ is different as a complex manifold (compare [BSW78, p. 238]). Hence, the $\bar{\partial}$-operators are different; so are the radii. The earlier works do not need to address the subtlety, but we have to, as it is crucial for our proof of the Greene-Wu conjectures.

To settle the subtlety, we derive in Lemma 12 [WY17] an interior gradient estimate for the solution of $\bar{\partial}$-equation. To get the estimate, we first transform the $\bar{\partial}$-equation into the Laplace equation for functions, in contrast to those standard Laplace equations for $(0,1)$-forms such as [FK72]. So, instead of the integral estimate of Morrey-Kohn-Hörmander, we apply the elementary maximum principle and Moser's iteration. Our approach has the advantage that the constants in the estimates clearly depend only on the curvature bounds. Thus, by the gradient estimate, we show that the radius can be chosen to depend only on the curvature bound. This justifies the effectiveness of quasi-bounded geometry.

Remark 3.5. In February 2019, Christina Sormani raised a question to us that whether there is a *harmonic* quasi-coordinate system on a complete Riemannian manifold of bounded curvature; that is, the quasi-coordinates are harmonic functions, the pullback metric under the quasi-coordinate map is uniformly equivalent to Euclidean metric, and the radii of quasi-coordinate balls and the constant in the metric equivalence depend only on the curvature bounds. The answer is affirmative. To see this, we again begin with the second author's Riemannian quasi-coordinate system $(B(R), \exp_x)$, where the radius $R > 0$ depends only on the curvature upper bound. Then, pullback the Riemannian metric to $B(R)$ via the exponential map. Then $B(R)$ with the pullback metric is itself a Riemannian manifold with the injectivity radius (of the origin) equal to $R > 0$. From here one can construct as usual the harmonic coordinates on a smaller ball $B(r)$; see for example, Theorem 2.8.1 in [Jos84, p. 59], where the radius $r > 0$ depends only on R and the curvature bounds. Hence, r depends only on the curvature bounds. □

Once the quasi-coordinate charts $\{(B(R_1), \psi_x)\}$ are obtained, one can define the Hölder spaces on M, as in Cheng-Yau [CY80], by pulling back the functions via ψ_x to $B(r)$ and taking supremum of all Hölder norms over $B(r)$. We can now adapt the Schauder theory to these Hölder spaces. This together with my generalized maximum principle enable us to prove the nonemptyness, openness, and the bootstrap argument.

Notice that in the process of constructing the quasi-coordinate chats, we assume that the background metric ω has bounds for all the covariant derivatives of its curvature tensor. To remove the derivative bounds, we invoke a useful derivative estimate, due to Wan-Xiong Shi [Shi89, Shi97]. Shi [Shi97] proves that if a complete Kähler manifold (M, ω) has bounded sectional curvature, then M admits another complete Kähler metric ω_1 which is uniformly equivalent to ω, and the curvature tensor of ω_1 has bounded covariant derivatives of arbitrary order. Shi's argument uses the Ricci flow and derive its short time existence. His argument can be extended to show that if the original metric ω has negatively pinched holomorphic sectional curvature, so is the new metric ω_1. An important step is to extend his maximum principle to tensors.

By replacing ω with ω_1, we can assume the curvature tensor of ω has bounded covariant derivatives of arbitrary order, in addition to $H(\omega)$ negatively pinched. Then, ω has quasi-bounded geometry. This settles the nonemptyness and openness of equation $\omega_t^n = e^u \omega^n$. By the refined Schwarz lemma we have $\mathrm{tr}_{\omega_t} \omega \leq C$. This together with the upper bound of u implies the closedness, where the bootstrap argument also uses the quasi-bounded geometry. This solves the Monge-Ampère type equation. Thus, we obtain a Kähler-Einstein metric ω_{KE} which is uniformly equivalent to ω. In particular, ω_{KE} is complete. The uniform estimates on u implies the curvature tensor of ω_{KE} has bounded covariant derivatives of any order. The uniqueness of ω_{KE} is already known (cf. [CY80, Proposition 5.5]), due to the second author's Schwarz Lemma [Yau78a]. This completes the proof of Theorem 3.1.

By using the (quasi-)bounded geometry with pointwise interior estimates, we can further prove the quasi-isometries of the Bergman metrics (Theorem 3.2) and Kobayashi-Royden metrics (Theorem 3.3). In fact, to show these two quasi-isometries we only need the effective (quasi-)bounded geometry of order zero. Thus, Shi's derivative estimates are not indispensable for these two theorems. We refer the readers to [WY17] for details.

Theorem 3.1 can be further generalized in several directions. One of them is connected to the study of the fourth classical invariant metric, the Carathéodory-Reiffen metric.

Theorem 3.6 ([WY18])**.** *Let (M, ω) be a complete Kähler manifold with bounded sectional curvature. Suppose that M has a holomorphic covering space $\tilde{M}$ such that for each point $x \in \tilde{M}$, there exists a holomorphic map F from $\tilde{M}$ to a Kähler manifold*

(N, ω_N) such that $H(\omega_N) \le -1$ and

$$F^*\omega_N \ge C\tilde{\omega} \quad \text{at } x$$

where $\tilde{\omega}$ is the induced covering metric, and $C > 0$ is a constant independent of x. Then, M admits a complete Kähler-Einstein metric ω_{KE} which is uniformly equivalent to ω and the curvature tensor of ω_{KE} and all its covariant derivatives are bounded on M.

Theorem 3.6 contains the previous theorem (Theorem 3.1), in that if the complete metric ω on M has negatively pinched holomorphic sectional curvature, then in particular its sectional curvature is bounded. We can simple take $\tilde{M} = N = M$ and $F =$ identity. We point out that the proof of Theorem 3.6 bypasses the maximum prinicple derived in [WY17, Appendix A], though the latter has interests of its own.

On the other hand, Theorem 3.6 can be viewed as an extension of the following result of H. Wu [Wu93, Theorem 2] (see also [Kik11, Corollary 1.2]) to the setting of complete Kähler manifold.

Theorem 3.7 ([Wu93, Kik11])**.** *Let M be a compact complex manifold. If M is Carathéodory hyperbolic, then K_M is ample.*

We include following result, which is technically different from Theorem 3.6, in order to include examples of the quasi-projective manifolds (compare for example [Yau78b, CY86, TY87, Wu08]).

Theorem 3.8. *Let (M, ω) be a complete Kähler manifold with bounded sectional curvature, and let $\pi : \tilde{M} \to M$ has a holomorphic covering space. Assume $\tilde{E} \subset \tilde{M}$ which is either compact or $\tilde{M} \setminus \tilde{E}$ is a bounded domain with respect to $\tilde{\omega} = \pi^*\omega$, such that*

(i) *$dd^c \log \tilde{\omega}^n \ge C_1 \tilde{\omega}$ on $\tilde{M} \setminus \tilde{E}$, where C_1 is a constant.*
(ii) *For each $x \in \tilde{E}$, there exists a holomorphic map F from $\tilde{M}$ to a Kähler manifold (N, ω_N) with $H(\omega_N) \le -1$ such that $F^*\omega_N \ge C_2 \tilde{\omega}$ where C_2 is a constant independent of x.*

Then, M admits a complete Kähler-Einstein metric ω_{KE} which is uniformly equivalent to ω, and the curvature tensor of ω_{KE} has bounded covariant derivatives of arbitrary order.

A motivational example for Theorem 3.6 and Theorem 3.8 is the moduli space of Riemann surfaces, whose covering space is the Teichülcer space (see [LSY04] for example). The Bers embedding theorem gives the map F from the covering space to a large ball in $\mathbb{C}^n$ so that the pullback metric under F is nondegenerate. Another example for Theorem 3.8 is the quasi-projective surface $M = \overline{M} \setminus D$ with positive logarithmic canonical bundle $K_{\overline{M}} + D$, where D is a Riemann surface of genus greater than one. Theorem 3.8 generates a new example of complete Kähler-Einstein manifold with infinite volume, modeled on $\mathbb{D} \times \mathbb{D}^*$; see [WY18, Example 5.5].

We note that Theorem 3.1 is recovered by Tong [Ton18] using the approach of Kähler-Ricci flow. Theorem 3.1 is generated by Huang-Lee-Tam-Tong [HLTT18] to the case with possibly unbounded curvature; namely, they replace condition $H(\omega) \geq -\kappa_2$ in Theorem 3.1 by the condition that there exists an smooth exhaustion function ρ with bounded gradient and bounded complex Hessian. It is then natural to ask if there is a geometric example of complete Kähler manifold (M, ω) with such an exhaustion function ρ, $H(\omega) \leq -\kappa_1 < 0$, and unbounded curvature.

4. Some open problems

We have mentioned several open problems, for instance, the conjectures of Lang, Kobayashi, and the second author in Section 2, and the two conjectures in Section 3 on the simply-connected complete Kähler manifold whose sectional curvature is bounded above by a negative constant. These conjectures are well-known. Concerning the second author's conjecture on compact hermitian manifolds, some progress has been made by [YZ19].

Very recently, F. Zheng [Zhe17] (see also [HLWZ18]) points out an interesting generalization of the second author's conjecture, based on the example of a smooth ample hypersurface in the abelian variety. The generalized conjecture states that if a compact Kähler manifold has nonpositive holomorphic sectional curvature and the curvature tensor has no truly flat direction at a point, then the canonical bundle is ample. (The curvature tensor is said to have a truly flat direction $v \in T'_x M$ at a point x if $R(v, \overline{\eta}, \xi, \overline{\zeta}) = 0$ for all $\eta, \xi, \zeta \in T'_x M$.)

It would be interesting to compare the three invariant metrics listed in Corollary 3.4 with the fourth classical invariant metric, the Carathéodory-Reffein metric, on a simply-connected complete Kähler manifold with negatively pinched curvature. The general existence of the last metric remains an open problem. For the bounded convex domain in $\mathbb{C}^n$, Lempert [Lem81] has shown that the Carathéodory-Reffein metric coincides with the Kobayashi-Royden metric. Very recently, by using the second author's Schwarz Lemma, G. Cho [Cho18a] compares the Carathéodory-Reffiein metric with the Kähler-Einstein metric on certain pseudoconvex domains.

Besides the above problems, we list below some questions from the viewpoints and methods developed in the previous sections.

Question 4.1. *Let (M, ω) be a compact Kähler manifold with quasi-negative holomorphic sectional curvature. Assume M contains no elliptic curve. Does M admit another Kähler metric ω_1 whose holomorphic sectional curvature is negative everywhere?*

This question is motivated from the purely partial differential equation approach in the proof of Theorem 2.3 (i). One might be able to deform ω to ω_1 along certain family of metrics which gradually decreases the holomorphic sectional curvature. The freeness of elliptic curve is imposed, in view of the Schwarz lemma. An answer to this question would allow one to gain a better understanding of the geometry of holomorphic curvature.

Conjecture 4.2. *Let M be a compact Kähler manifold such that over the projectivized tangent bundle $\mathbb{P}(T'M)$ of M, the first Chern class of the tautological line bundle is negative in the tautological direction. Then, the canonical bundle K_M is positive.*

Conjecture 4.2 may be viewed as an algebraic geometric generalization or a Finsler generalization of [WY16a]. To see this, if M has a Kähler metric ω whose holomorphic sectional curvature is negative, then ω naturally induces a metric on the tautological line bundle such that its curvature form is negative along the tautological direction. On the other hand, the negativity of the first Chern class in Conjecture 4.2 has an algebraic geometric interpretation, which does not require the notion of curvature.

Conjecture 4.3. *Let (M, ω) be a simply-connected complete Kähler manifold with sectional curvature bounded between two negative constants. Then M is biholomorphic to a bounded pseudoconvex domain in $\mathbb{C}^n$.*

Conjecture 4.3 may be viewed as an intermediate step toward the conjecture in Section 3, which asserts that a simply-connected complete Kähler manifold with sectional curvature bounded above by a negative constant must be biholomorphic to a bounded domain in $\mathbb{C}^n$. For, if the latter conjecture is true, then the bounded domain in $\mathbb{C}^n$ possesses a complete Kähler-Einstein metric, by virtue of Theorem 3.1. It then follows from [MY83] that the bounded domain is psuedoconvex. We note that the pseudoconvex cannot be strengthened to the strict pseudoconvexity, in view of the egg domain [Bla86] (see also [Cho18b]).

At this end, let us discuss another open problem which has been posted by the second author a while ago. This is a question about resolution of singularities of Kähler metrics. Let us look at the following class of metrics: Take a complex variety M and a subvariety S of M, we consider Kähler metrics g defined in $M - S$ that satisfies the following condition: At each point $x \in S$, there is a neighborhood U of x so that a nonsingular manifold O and a subvariety D of O and a holomorphic map $F : O \to U$ which maps D into S so that each component of the inverse of $S \cap U$ is a compact subvariety of D. (In fact, a component of the inverse image of a compact neighborhood is compact.)

The map is locally invertible on every point in $O - D$, and the pullback of the metric g (defined on $M - S$) under F can be extended to be a smooth nonsingular metric on O. We also allow the pullback metric to be a Kähler metric defined on $O - D$, complete towards D and its curvature and covariant derivatives are bounded.

The Kähler metric g is said to *admit resolution of singularities* if a system of maps $\{F\}$ exists at every point $x \in S$. A good example is the orbifold metric where O can be taken to be the ball and the map F is the map from the ball to its quotient space which maps the origin to the quotient singularity. Note that the singular behavior of the metric depends on the system $\{F\}$ which is defined in holomorphic category.

The second author conjectures that *if the curvature and the covariant derivatives of the curvature of this Kähler metric are bounded in each neighborhood of x of S, the resolution system $\{F\}$ exists*. Such a statement may be called *resolution of singularities of Kähler metric*. Note that if we fix a holomorphic system $\{F\}$, there is only one complete Kähler-Einstein metric with negative Ricci curvature resolved by $\{F\}$.

On the other hand, there can be distinct Kähler-Einstein metrics if we choose different system to resolve the singularities of the metric. One can define two systems of resolutions to be equivalent if the holomorphic map from $O - D$ to $O' - D'$ can be extended to be a nonsingular map from O to O' and the same is true for the inverse map from $O' - D'$ to $O - D$.

This concept appeared in the work of the second author with Cheng [CY86] on the construction of Kähler-Einstein metrics on singular varieties. The existence of Kähler-Einstein metrics can be readily generalized to this class of singular metrics. (Basically the same argument in [CY86].)

Is it true that algebraic manifolds of general type admits such Kähler metrics with negative Ricci curvature? It is certainly true for algebraic surface of general type. Since the arguments of Ricci flow largely depend only on maximal principle, Kähler-Ricci flow works well with class of singular metrics. Note that such Kähler metric includes a class of Kähler metrics which can be degenerate along S.

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